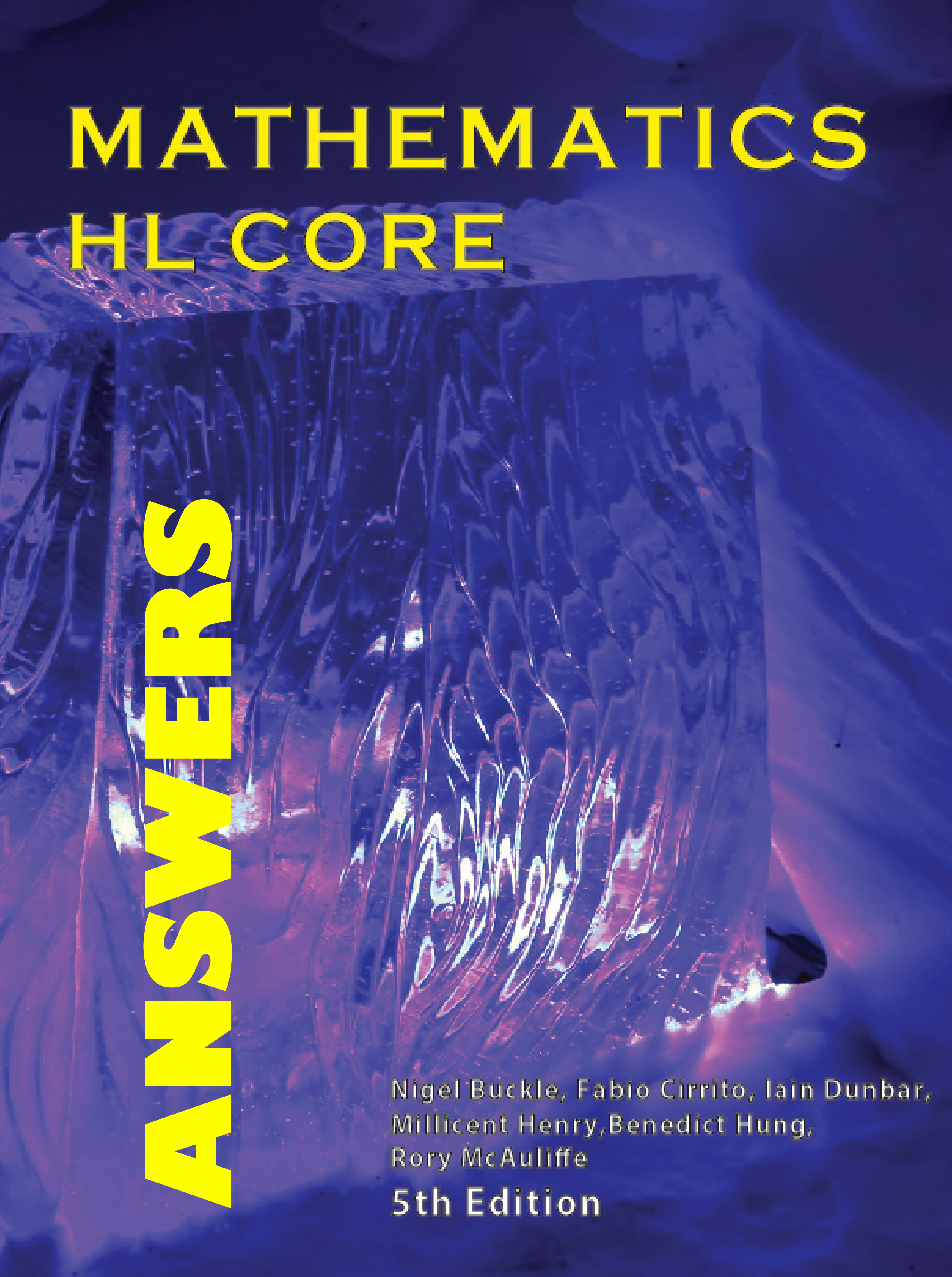
MATHEMATICS HL CORE

ANSWERS



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5th Edition

FOR USE WITH THE I.B. DIPLOMA PROGRAMME

Exercise 1.1.1

1.	**i**	**b**	4	**c**	$t_n = 4n-2$
	ii	**b**	−3	**c**	$t_n = -3n+23$
	iii	**b**	−5	**c**	$t_n = -5n+6$
	iv	**b**	0.5	**c**	$t_n = 0.5n$
	v	**b**	2	**c**	$t_n = \mathrm{y}+2n-1$
	vi	**b**	−2	**c**	$t_n = x-2n+4$

2 −28

3 9,17

4 −43

5 7

6 7

7 −5

8 0

9 **a** 41 **b** 31st

10 $2, \sqrt{3}$

11 **a i** 2 **ii** −3 **b i** 4 **ii** 11

12 $x-8y$

13 $t_n = 5+\frac{10}{3}(n-1)$

14 **a** −1 **b** 0

Exercise 1.1.2

1	**a**	145	**b**	300	**c**	−170
2	**a**	−18	**b**	690	**c**	70.4
3	**a**	−105	**b**	507	**c**	224
4	**a** 126		**b**	3900	**c**	14th week
5	855					
6	**a**	420	**b**	−210		
7	$\mathrm{a} = 9, \mathrm{b} = 7$					

Exercise 1.1.3

1 123

2 –3, –0.5, 2, 4.5, 7, 9.5, 12

3 3.25

4 $a = 3 \quad d = -0.05$

5 10 000

6 330

7 –20

8 328

9 \$725, 37 weeks

10 **a** \$55 **b** 2750

11 **a i** 8 m **ii** 40 m **b** 84 m

c Dist = $2n^2 - 2n = 2n(n-1)$

d 8 **e** 26 players, 1300 m

12 **a** 5050 **b** 10200 **c** 4233

13 **a** 145 **b** 390 **c** –1845

14 **b** $3n - 2$

Exercise 1.1.4

1 **a** $r = 2, u_5 = 48, u_n = 3 \times 2^{n-1}$

b $r = \frac{1}{3}, u_5 = \frac{1}{27}, u_n = 3 \times \left(\frac{1}{3}\right)^{n-1}$

c $r = \frac{1}{5}, u_5 = \frac{2}{625}, u_n = 2 \times \left(\frac{1}{5}\right)^{n-1}$

d $r = -4, u_5 = -256, u_n = -1 \times (-4)^{n-1}$

e $r = \frac{1}{b}, u_5 = \frac{a}{b^3}, u_n = ab \times \left(\frac{1}{b}\right)^{n-1}$

f $r = \frac{b}{a}, u_5 = \frac{b^4}{a^2}, u_n = a^2 \times \left(\frac{b}{a}\right)^{n-1}$

2 **a** ±12 **b** $\frac{\pm\sqrt{5}}{2}$

3 **a** ±96 **b** 15th

4 **a** $u_n = 10 \times \left(\frac{5}{6}\right)^{n-1}$ **b** $\frac{15625}{3888} \cong 4.02$ **c** $n = 5$ 4 times

5 $-2, \frac{4}{3}$

6 **a i** \$4096 **ii** \$2097.15 **b** 6.2 yrs

7 $\left(u_n = \frac{1000}{169} \times \left(\frac{12}{5}\right)^{n-1}\right), \frac{1990656}{4225} \cong 471.16$

8 2.5,5,10 or 10,5,2.5

9 53 757

10 108 952

11 **a** \$56 156 **b** \$299 284

Exercise 1.1.5

1 **a** 3 **b** $\frac{1}{3}$ **c** −1 **d** $-\frac{1}{3}$ **e** 1.25
f $-\frac{2}{3}$

2 **a** 216513 **b** 1.6384×10^{-10} **c** $\frac{256}{729}$
d $\frac{729}{2401}$ **e** $-\frac{81}{1024}$

3 **a** 11; 354 292 **b** 7; 473 **c** 8; 90.90909
d 8; 172.778 **e** 5; 2.256 **f** 13; 111.1111111111

4 **a** $\frac{127}{128}$ **b** $\frac{63}{8}$ **c** $\frac{130}{81}$
d 60 **e** $\frac{63}{64}$

5 4; 118 096

6 \$2109.50

7 9.28 cm

8 **a** $V_n = V_0 \times 0.7^n$ **b** 7

9 54

10 53.5 gms; 50 weeks.

11 7

12 9

13 −0.5, −0.7797

14 $r = 5$, 1.8×10^{10}

15 $8407.35

16 1.8×10^{19} or about 200 billion tonnes.

Exercise 1.1.6

1 Term 9 AP = 180, GP = 256. Sum to 11 terms AP = 1650, GP = 2047.

2 18

3 12

4 7, 12

5 8 weeks Ken $220 & Bo-Youn $255)

6 **a** week 8 **b** week 12

7 **a** 1.618

b 121 379 [~121400, depends on rounding errors]

Exercise 1.1.7

1 **a** $\frac{81}{2}$ **b** $\frac{10}{13}$ **c** 5000 **d** $\frac{30}{11}$

2 $23\frac{23}{99}$

3 6667 fish. [NB: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish.

Overfishing means that fewer fish are caught in the long run.

4 27

5 48,12,3 or 16,12,9

6 **a** $\frac{11}{30}$ **b** $\frac{37}{99}$ **c** $\frac{191}{90}$

7 128 cm

8 $\frac{121}{9}$

9 $2 + \frac{4}{3}\sqrt{3}$

10 $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$

11 $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

Exercise 1.1.8

1 3, –0.2

2 $\frac{2560}{93}$

3 $\frac{10}{3}$

4 **a** $\frac{43}{18}$ **b** $\frac{458}{99}$ **c** $\frac{413}{990}$

5 9900

6 3275

7 3

8 $t_n = 6n - 14$

9 6

10 $-\frac{1}{6}$

11 **a** 12 **b** 26

12 9, 12

13 ±2

14 (5, 5, 5), (5, –10, 20)

15 **a** 2, 7 **b** 2, 5, 8 **c** $3n - 1$

16 **a** 5 **b** 2 m

Exercise 1.1.9

1 $2773.08

2 $4377.63

3 $1781.94

4 $12 216

5 $35 816.95

6 $40 349.37

7 $64 006.80

8 $276 971.93, $281 325.41

9 \$63 762.25

10 \$98.62, \$9467.14, interest \$4467.14. Flat interest = \$6000

11 \$134.41, \$3790.44, 0.602% /month (or 7.22% p.a.)

12 –½, 3 The sequence ½, ½, ½,.. is arithmetic.

13 15

14 Proof

15 $m = 19$, $n = 34$

Exercise 1.2.1

1 Simplify the following.

e $\left(\frac{2x^3}{4y^2}\right)^2 \times \frac{12y^6}{8x^4}$ f $\frac{3^{n+2}+9}{3}$ g $\frac{4^{n+2}-16}{4}$ h $\frac{4^{n+2}-16}{2}$

i $\left(\frac{1}{2b}\right)^4 - \frac{b^2}{16}$

2 Simplify the following.

e $\frac{(xy)^6}{64x^6}$ f $\frac{27^{n+2}}{6^{n+2}}$

3 Simplify the following.

e $\frac{2^n \times 4^{2n+1}}{2^{1-n}}$ f $\frac{2^{2n+1} \times 4^{-n}}{(2^n)^3}$ g $\frac{x^{4n+1}}{(x^{n+1})^{(n-1)}}$ h $\frac{x^{4n^2+n}}{(x^{n+1})^{(n-1)}}$

i $\frac{(3^x)(3^{x+1})(3^2)}{(3^x)^2}$

5 Simplify the following, leaving your answer in positive power form.

e $\frac{(-2)^3 \times 2^{-3}}{(x^{-1})^2 \times x^2}$ f $\frac{(-a)^3 \times a^{-3}}{(b^{-1})^{-2}b^{-3}}$

6 Simplify the following.

e $\frac{(x-1)^{-3}}{(x+1)^{-1}(x^2-1)^2}$ f $\frac{y(x^{-1})^2 + x^{-1}}{x+y}$

7 Simplify the following.

a $5^{n+1} - 5^{n-1} - 2 \times 5^{n-2}$ b $a^{x-y} \times a^{y-z} \times a^{z-x}$ c $\left(\frac{a^{-\frac{1}{2}}b^3}{ab^{-1}}\right)^2 \times \frac{1}{ab}$

d $\left(\frac{a^{m+n}}{a^n}\right)^m \times \left(\frac{a^{n-m}}{a^n}\right)^{m-n}$ e $\frac{p^{-2}-q^{-2}}{p^{-1}-q^{-1}}$ f $\frac{1}{1+a^{\frac{1}{2}}} - \frac{1}{1-a^{\frac{1}{2}}}$

g $\frac{2^{n+4}-2(2^n)}{2(2^{n+3})}$ h $\sqrt{a\sqrt{a\sqrt{a}}}$

8 Simplify the following.

a $\frac{\sqrt{x} \times \sqrt[3]{x^2}}{\sqrt[4]{x}}$ b $\frac{b^{n+1} \times 8a^{2n-1}}{(2b)^2(ab)^{-n+1}}$ c $\frac{2^n - 6^n}{1-3^n}$

d $\frac{7^{m+1}-7^m}{7^n-7^{n+2}}$ e $\frac{5^{2n+1}+25^n}{5^{2n}+5^{1+n}}$

Exercise 1.2.3

1 Use the definition of a logarithm to determine the following.

g $\log_4 1$ h $\log_{10} 1$ i $\log_{\frac{1}{2}} 2$ j $\log_{\frac{1}{3}} 9$

k $\log_3 \sqrt{3}$ l $\log_{10} 0.01$

3 Change the following logarithmic expression into its equivalent exponential form.

f $\log_2(ax - b) = y$

4 Solve for x in each of the following.

g $\log_x 16 = 2$ h $\log_x 81 = 2$

i $\log_x\left(\frac{1}{3}\right) = 3$ j $\log_2(x-5) = 4$

k $\log_3 81 = x + 1$ l $\log_3(x-4) = 2$

5 Solve for x in each of the following, giving your answer to 4 d.p.

g $\log_e(x+2) = 4$ h $\log_e(x-2) = 1$ i $\log_x e = -2$

Exercise 1.2.4

1 Without using a calculator, evaluate the following.

e $\log_2 20 - \log_2 5$ f $\log_2 10 - \log_2 5$

2 Write down an expression for $\log a$ in terms of $\log b$ and $\log c$ for the following.

e $a = b^3 c^4$ f $a = \frac{b^2}{\sqrt{c}}$

4 Express each of the following as an equation that does not involve a logarithm.

d $\log_2 x = y + 1$ e $\log_2 y = \frac{1}{2}\log_2 x$ f $3\log_2(x+1) = 2\log_2 y$

5 Solve the following equations.

d $\log_{10}(x+3) - \log_{10} x = \log_{10} x + \log_{10} 2$

e $\log_{10}(x^2+1) - 2\log_{10} x = 1$

f $\log_2(3x^2+28) - \log_2(3x-2) = 1$

g $\log_{10}(x^2+1) = 1 + \log_{10}(x-2)$

h $\log_2(x+3) = 1 - \log_2(x-2)$

i $\log_6(x+5) + \log_6 x = 2$

j $\log_3(x-2) + \log_3(x-4) = 2$

k $\log_2 x - \log_2(x-1) = 3\log_2 4$

l $\log_{10}(x+2) - \log_{10} x = 2\log_{10} 4$

6 Simplify the following

c $2\log_a x + 3\log_a(x+1)$ d $5\log_a x - \frac{1}{2}\log_a(2x-3) + 3\log_a(x+1)$

e $\log_{10} x^3 + \frac{1}{3}\log x^3 y^6 - 5\log_{10} x$ f $2\log_2 x - 4\log_2\left(\frac{1}{y}\right) - 3\log_2 xy$

7 Solve the following

d $\log_3 x + \log_3(x-8) = 2$

e $\log_2 x + \log_2 x^3 = 4$

f $\log_3\sqrt{x} + 3\log_3 x = 7$

8 Solve for x.

c $\log_4 x^4 = (\log_4 x)^4$ d $\log_5 x^5 = (\log_5 x)^5$

e Investigate the solution to $\log_n x^n = (\log_n x)^n$.

9 Solve the following, giving an exact answer and an answer to 2 d.p.

e $3^{4x+1} = 10$ f $0.8^{x-1} = 0.4$ g $10^{-2x} = 2$

h $2.7^{0.3x} = 9$ i $0.2^{-2x} = 20$ j $\frac{2}{1+0.4^x} = 5$

k $\frac{2^x}{1-2^x} = 3$ l $\frac{3^x}{3^x+3} = \frac{1}{3}$

10 Solve for x.

c $\log_{10}(x^2-3x+6) = 1$ d $(\log_{10}x)^2 - 11\log_{10}x + 10 = 0$

e $\log_x(3x^2+10x) = 3$ f $\log_{x+2}(3x^2+4x-14) = 2$

11 Solve the following simultaneous equations.

c
$$xy = 2$$
$$2\log_2 x - \log_2 y = 2$$

12 Express each of the following as an equation that does not involve a logarithm.

c $\ln x = y - 1$

13 Solve the following for x.

c $\log_e(x+1) + \log_e x = 0$ d $\log_e(x+1) - \log_e x = 0$

14 Solve the following for x.

c $-5 + e^{-x} = 2$ d $200e^{-2x} = 50$ e $\frac{2}{1-e^{-x}} = 3$

f $70e^{-\frac{1}{2}x} + 15 = 60$ g $\ln x = 3$ h $2\ln(3x) = 4$

i $\ln(x^2) = 9$ j $\ln x - \ln(x+2) = 3$ k $\ln\sqrt{x+4} = 1$ l $\ln(x^3) = 9$

15 Solve the following for x.

c $e^{2x} - 5e^{x} + 6 = 0$

d $e^{2x} - 2e^{x} + 1 = 0$

e $e^{2x} - 6e^{x} + 5 = 0$

f $e^{2x} - 9e^{x} - 10 = 0$

16 Solve each of the following.

a $4^{x-1} = 132$

b $5^{5x-1} = 3^{1-2x}$

c $3^{2x+1} - 7 \times 3^{x} + 4 = 0$

d $2^{2x+3} - 7 \times 2^{x+1} + 5 = 0$

e $3 \times 4^{2x+1} - 2 \times 4^{x+2} + 5 = 0$

f $3^{2x} - 3^{x+2} + 8 = 0$

g $2\log x + \log 4 = \log(9x - 2)$

h $2\log 2x - \log 4 = \log(2x - 1)$

i $\log_3 2x + \log_3 81 = 9$

j $\log_2 x + \log_x 2 = 2$

Exercise 1.2.1

1	a	$\frac{27y^{15}}{8x^3}$	b	$\frac{91}{216a^6}$	c	$2^n + 2$	d	$\frac{8x^{11}}{27y^2}$
	e	$\frac{3x^2y^2}{8}$	f	$3^{n+1} + 3$	g	$4^{n+1} - 4$		
	h	$2(4^{n+1} - 4)$	i	$\frac{1-b^6}{16b^4}$				
2	a	64	b	$\left(\frac{2}{3}\right)^x$	c	2^{2y+1}	d	$\frac{1}{b^{2x}}$
	e	$\left(\frac{y}{2}\right)^6$	f	$\left(\frac{9}{2}\right)^{n+2}$				
3	a	$\frac{z^2}{xy}$	b	3^{7n-2}	c	5^{n+1}	d	9
	e	2^{6n+1}	f	2^{1-3n}	g	x^{2+4n-n^2}		
	h	x^{3n^2+n+1}	i	27				
4	$\frac{y^{2m-2}}{x^m}$							
5	a	-81	b	$-\frac{9x^8}{8y^4}$	c	$y - x$		
	d	$\frac{2x+1}{x+1}$	e	-1	f	$-b$		
6	a	$\frac{1}{x^2y^2}$	b	$\frac{1}{x^4}$	c	$-\frac{1}{x(x+h)}$		
	d	$\frac{1}{x-1}$	e	$\frac{1}{(x+1)(x-1)^5}$	f	$\frac{1}{x^2}$		
7	a	$118 \times 5^{n-2}$	b	1	c	$\frac{b^7}{a^4}$	d	a^{mn}
	e	$\frac{p+q}{pq}$	f	$\frac{2\sqrt{a}}{a-1}$	g	$\frac{7}{8}$	h	$a^{7/8}$
8	a	$x^{11/12}$	b	$2a^{3n-2}b^{2n-2}$	c	2^n		
	d	$-\frac{7^{m-n}}{8}$	e	$\frac{6 \times 5^n}{5^n + 5}$				

Exercise 1.2.2

1	a	2	b	−2	c	$\frac{2}{3}$	d	5	e	6
	f	−2.5	g	2	h	1.25	i	$\frac{1}{3}$		
2	a	−6	b	$-\frac{2}{3}$	c	−3	d	1.5	e	0.25
	f	0.25	g	$-\frac{1}{8}$	h	$-\frac{11}{4}$	i	−1.25		

Exercise 1.2.3

1	a	2	b	2	c	5	d	3	e	−3
	f	−2	g	0	h	0	i	−1	j	−2
	k	0.5	l	−2						

2 **a** $\log_{10}10000 = 4$ **b** $\log_{10}0.001 = -3$
c $\log_{10}(x+1) = y$ **d** $\log_{10}p = 7$
e $\log_2(x-1) = y$ **f** $\log_2(y-2) = 4x$

3 **a** $2^9 = x$ **b** $b^x = y$ **c** $b^{ax} = t$
d $10^{x^2} = z$ **e** $10^{1-x} = y$ **f** $2^y = ax - b$

4 **a** 16 **b** 2 **c** 2 **d** 9 **e** $\sqrt[4]{2}$
f 125 **g** 4 **h** 9 **i** $\sqrt[3]{\frac{1}{3}}$ **j** 21 **k** 3
l 13

5 **a** 54.5982 **b** 1.3863 **c** 1.6487
d 7.3891 **e** 1.6487 **f** 0.3679
g 52.5982 **h** 4.7183 **i** 0.6065

Exercise 1.2.4

1 **a** 5 **b** 2 **c** 2 **d** 1 **e** 2 **f** 1

2 **a** $\log a = \log b + \log c$ **b** $\log a = 2\log b + \log c$
c $\log a = -2\log c$ **d** $\log a = \log b + 0.5\log c$
e $\log a = 3\log b + 4\log c$ **f** $\log a = 2\log b - 0.5\log c$

3 **a** 0.18 **b** 0.045 **c** –0.09

4 **a** $x = yz$ **b** $y = x^2$ **c** $y = \frac{x+1}{x}$
d $x = 2^{y+1}$ **e** $y = \sqrt{x}$ **f** $y^2 = (x+1)^3$

5 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{17}{15}$ **d** $\frac{3}{2}$ **e** $\frac{1}{3}$
f no real soln **g** 3,7 **h** $\frac{\sqrt{33}-1}{2}$ **i** 4
j $\sqrt{10}+3$ **k** $\frac{64}{63}$ **l** $\frac{2}{15}$

6 **a** $\log_3 2wx$ **b** $\log\frac{x}{47y}$ **c** $\log_a[x^2(x+1)^3]$
d $\log_a\left[\frac{(x^5)(x+1)^3}{\sqrt{2x-3}}\right]$ **e** $\log_{10}\left(\frac{y^2}{x}\right)$ **f** $\log_2\left(\frac{y}{x}\right)$

7 **a** 1 **b** –2 **c** 3 **d** 9 **e** 2 **f** 9

8 **a** 1,4 **b** $1, 3^{\pm\sqrt{3}}$ **c** $1, 4^{\sqrt[3]{4}}$ **d** $1, 5^{\pm\sqrt[4]{5}}$

9 **a** $\frac{\log 14}{\log 2} = 3.81$ **b** $\frac{\log 8}{\log 10} = 0.90$ **c** $\frac{\log 125}{\log 3} = 4.39$
d $\frac{1}{\log 2} \times \log\left(\frac{11}{3}\right) - 2 = -0.13$ **e** $\frac{\log 10 - \log 3}{4\log 3} = 0.27$
f 5.11 **g** $\frac{-\log 2}{2\log 10} = -0.15$

	h	7.37	**i**	0.93	**j**	no real solution				
	k	$\frac{\log 3}{\log 2} - 2 = -0.42$			**l**	$\frac{\log 1.5}{\log 3} = 0.37$				
10	**a**	0.5,4	**b**	3	**c**	−1,4	**d**	$10, 10^{10}$	**e**	5
	f	3								
11	**a**	$(4, \log_4 11)$	**b**	100,10	**c**	2,1				
12	**a**	$y = xz$	**b**	$y = x^3$	**c**	$x = e^{y-1}$				
13	**a**	$\frac{1}{e^4 - 1}$	**b**	$\frac{1}{3}$	**c**	$\frac{\sqrt{5} - 1}{2}$	**d**	$\varnothing$		
14	**a**	$\ln 21 = 3.0445$	**b**	$\ln 10 = 2.3026$	**c**	$-\ln 7 = -1.9459$				
	d	$\ln 2 = 0.6931$	**e**	$\ln 3 = 1.0986$						
	f	$2\ln\left(\frac{14}{9}\right) = 0.8837$	**g**	$e^3 = 20.0855$						
	h	$\frac{1}{3}e^2 = 2.4630$	**i**	$\pm\sqrt{e^9} = \pm 90.0171$	**j**	$\varnothing$				
	k	$e^2 - 4 = 3.3891$	**l**	$\sqrt[3]{e^9} = 20.0855$						
15	**a**	$0, \ln 2$	**b**	$\ln 5$	**c**	$\ln 2, \ln 3$	**d**	0		
	e	$0, \ln 5$	**f**	$\ln 10$						
16	**a**	4.5222	**b**	0.2643	**c**	0,0.2619				
	d	−1,0.3219	**e**	−1.2925,0.6610	**f**	0,1.8928				
	g	0.25,2	**h**	1	**i**	121.5	**j**	2		

Exercise 1.3.2

9. In how many ways can 3 boys and 2 girls be arranged in a row if a selection is made from 6 boys and 5 girls?

10. If $\binom{n}{3} = 56$ show that $n^3 - 3n^2 + 2n - 336 = 0$. Hence find n.

11. In how many ways can a jury of 12 be selected from 9 men and 6 women so that there are at least 6 men and no more than 4 women on the jury.

12. Show that $\binom{n+1}{3} - \binom{n-1}{3} = (n-1)^2$. Hence find n if $\binom{n+1}{3} - \binom{n-1}{3} = 16$.

Exercise 1.3.3

18. In how many ways can 4 people be accommodated if there are 4 rooms available?

19. A car can hold 3 people in the front seat and 4 in the back seat. In how many ways can 7 people be seated in the car if John and Samantha must sit in the back seat and there is only one driver?

20. In how many ways can six men and two boys be arranged in a row if:

    a the two boys are together?

    b the two boys are not together?

    c there are at least three men separating the boys?

21. In how many ways can the letters of the word "TOGETHER" be arranged? In how many of these arrangements are all the vowels together?

22. In how many ways can 4 women and 3 men be arranged in a row, if there are 8 women and 5 men to select from?

23. In how many ways can 4 women and 3 men be arranged in a circle? In how many ways can this be done if the tallest woman and shortest man must be next to each other?

24. In how many ways can 5 maths books, 4 physics books and 3 biology books be arranged on a shelf if subjects are kept together?

25. How many even numbers of 4 digits can be formed using 5, 6, 7, 8 if:

    a no figure is repeated?

    b repetition is allowed?

16. Five men and 5 women are to be seated around a circular table. In how many ways can this be done if the men and women alternate?

27. A class of 20 students contains 5 student representatives. A committee of 8 is to be formed. How many different committees can be formed if there are:

    a only 3 student representatives?

    b at least 3 student representatives?

28. How many possible juries of 12 can be selected from 12 women and 8 men so that there are at least 5 men and not more than 7 women?

29. In how many ways can 6 people be seated around a table if 2 friends are always:

    a together?

    b separated?

Exercise 1.3.1

1	15							
2	a	25	b	625				
3	a	24	b	256				
4	a	24	b	48				
5	15							
6	270							
7	120							
8	336							
9	60							
10	a	362880	b	80640	c	1728		
11	20							
12	a	10!	b	2×8!	c i 2×9!		ii 8×9!	
13	34650							
14	4200							
15	4							

Exercise 1.3.2

1	792			
2	a	1140	b	171
3	1050			
4	70			
5	2688			
6	a	210	b	420
7	889			
9	24000			
10	8			
11	155			
12	5			

Exercise 1.3.3

1	a	120	b	325
2	5040			
3	a	144	b	1440
4	a	720	b	240
5	11760			
6	7056; 4606			
7	a	840	b	1680
8	190			
9	10080			
10	226800			
11	a 71 b 315 c 665			
13	${}^{n}C_2$			
14	${}^{n}C_4$			

15 b 92
16 252
17 a 1287 b 560
18 256
19 288
20 a 10 080 b 30 240 c 14 400
21 10 080, 1080
22 3 528 000
23 720; 240
24 103 680
25 a 12 b 128
26 2880
27 a 30 030 b 37 310
28 77 055
29 a 48 b 72

Exercise 1.3.4

1 a $b^2 + 2bc + c^2$ b $a^3 + 3a^2g + 3ag^2 + g^3$

c $1 + 3y + 3y^2 + y^3$ d $16 + 32x + 24x^2 + 8x^3 + x^4$

e $8 + 24x + 24x^2 + 8x^3$ f $8x^3 - 48x^2 + 96x - 64$

g $16 + \frac{32}{7}x + \frac{24}{49}x^2 + \frac{8}{343}x^3 + \frac{1}{2401}x^4$ h $8x^3 - 60x^2 + 150x - 125$

i $27x^3 - 108x^2 + 144x - 64$ j $27x^3 - 243x^2 + 729x - 729$

k $8x^3 + 72x^2 + 216x + 216$ l $b^3 + 9b^2d + 27bd^2 + 27d^3$

m $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

n $x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

o $\frac{125}{p^3} + \frac{150}{p} + 60p + 8p^3$ p $\frac{16}{x^4} - \frac{32}{x} + 24x^2 - 8x^5 + x^8$

q $q^5 + \frac{10q^4}{p^3} + \frac{40q^3}{p^6} + \frac{80q^2}{p^9} + \frac{80q}{p^{12}} + \frac{32}{p^{15}}$

r $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

Exercise 1.3.5

1 a $160x^3$ b $21x^5y^2$ c $-448x^3$

d $-810x^4$ e $216p^4$

f $-20412p^2q^5$ g $-22680p$

2	a	-1400000	b	6000	c	540
	d	-240	e	81648	f	40

3 1.0406 0.0004%

4 a $64x^6 + 960x^5 + 6000x^4 + 20000x^3 + 37500x^2 + 37500x + 15625$

b 19750 c 20.6 d 0.1%

5 19

6 $-\frac{63}{8}$

7 $\frac{231}{16}$

8 $-\frac{130}{27}$

9 -20

10 $a = \pm 3$

11 $n = 5$

12 $n = 9$

13 a 0 b -59

14 $a = 3, n = 8$

15 $a = \pm 2, b = \pm 1$

16. a $a = -3$ $n = 6$

b $a = {}^1/_3$ $n = 5$

c $a = \sqrt{2}$ $n = 4$

d $a = -\dfrac{\sqrt{2}}{6}$ $n = 4$

Proof of the Binomial Theorem

Note: This proof by induction is outside the scope of the syllabus.

A formal statement of the binomial expansion is:

$$(a+b)^k = {}^nC_0a^kb^0 + {}^nC_1a^{k-1}b^1 + \dots + {}^nC_ra^{k-r}b^r + \dots + {}^nC_ka^0b^k = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

The binomial theorem for positive integral index may be proved using mathematical induction.

A preliminary result from combinatorics is required, namely ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$. We leave its proof as an exercise. We can now move on to the main induction proof:

Check the case $k = 1$: $(a+b)^1 = {}^1C_0a^1 + {}^1C_0b^1 = a+b$, which is true.

Assume the theorem is true for $n = k$:

$$(a+b)^k = {}^kC_0a^kb^0 + {}^kC_1a^kb^1 + \dots + {}^kC_ra^{k-r}b^r + \dots + {}^kC_ka^0b^k$$

Look at the $n = k + 1$. This involves multiplying each term in the expansion from step **2** first by a and then by b. To see what happens to the general term, it is a good idea to look at two consecutive terms in the middle of the expansion from step **2**:

$$(a+b)^k = {}^kC_0a^kb^0 + \dots + {}^kC_{r-1}a^{k-r+1}b^{r-1} + {}^kC_ra^{k-r}b^r + \dots + {}^kC_ka^0b^k$$

When this expansion has been multiplied by a, the result is:

$${}^kC_0a^{k+1}b^0 + \dots + {}^kC_{r-1}a^{k-r+2}b^{r-1} + {}^kC_ra^{k-r+1}b^r + \dots + {}^kC_ka^1b^k$$

and when it is multiplied by b, the result is:

$${}^kC_0a^kb^1 + \dots + {}^kC_{r-1}a^{k-r+1}b^r + {}^kC_ra^{k-r}b^{r+1} + \dots + {}^kC_ka^0b^{k+1}$$

The expansion of $(a+b)^{k+1}$ begins with ${}^kC_0a^{k+1}b^0 = a^{k+1}b^0 = a^{k+1}$, which is correct.

The expansion ends with ${}^kC_ka^0b^{k+1} = a^0b^{k+1} = b^{k+1}$, which is also correct.

It now remains to prove that the general term in the middle of the expansion is also correct. Lining up like terms from the two parts of the expansion gives:

$${}^kC_{r-1}a^{k-r+2}b^{r-1} + {}^kC_ra^{k-r+1}b^r$$

$${}^kC_{r-1}a^{k-r+1}b^{r-1} + {}^kC_ra^{k-r}b^{r+1}$$

The general term is: ${}^kC_ra^{k-r+1}b^r + {}^kC_{r-1}a^{k-r+1}b^r = [{}^kC_r + {}^kC_{r-1}](a^{k-r+1}b^r)$

$$= {}^{k+1}C_ra^{k-r+1}b^r$$ (using the combinatorial result given at the start of this section)

We can conclude that the binomial theorem gives the correct expansion for $n = 1$ from part **i**. Part **iii** indicates that the theorem gives the correct expansion for an index of 2, 3 etc. Hence the theorem holds for all positive integral indices.

Exercise 1.4.2

By induction, prove that:

g $n + 1 < 3n$ for all $n \geq 1$

h $1 + n^2 < (1 + n)^2$ for all $n \geq 1$

i $n < 1 + n^2$ for all $n \geq 1$

j $7^n - 3^n$, $n \geq 1$ is divisible by 4

[Hint: $7^{k+1} - 3^{k+1} = (7^{k+1} - 7\times3^k) + (7\times3^k - 3^{k+1})$]

Exercise 1.4.3

Prove the following using the principle of mathematical induction for all $n \in \mathbb{Z}^+$.

i $1.3.5 + 2.4.6 + \ldots + n(n+2)(n+4) = \frac{1}{4}n(n+1)(n+4)(n+5)$

j $\frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$

k $\frac{1}{1.3} + \frac{1}{2.4} + \ldots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

l $(1+x)^n > 1 + nx + nx^2$ for all $n \geq 3, x > 0$

m $2^n \geq n^2$ for all integers $n \geq 4$.

n $1.1! + 2.2! + 3.3! + \ldots + n.n! = (n+1)! - 1$

o $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$

p $n! > n^2$ for all $n > 3$.

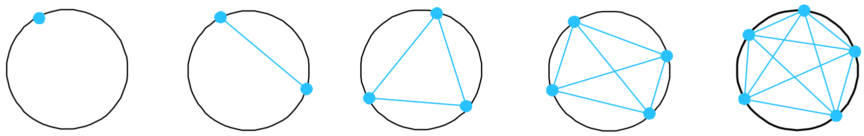
q $n^5 - n$ is divisible by 5.

r $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$, $n \geq 1$.

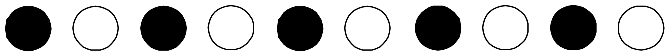
s Prove that the maximum number of points of intersection of $n \geq 2$ lines in a plane is $\frac{1}{2}n(n-1)$.

Exercise 1.4.4

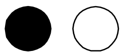
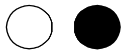
8. In each of these circles, each dot is joined to every other one. Into how many pieces is each circle divided? If a circle is drawn with n dots, what is the maximum number of regions that can be formed if all dots are joined in the same manner?



9. You are given five black discs and five white discs which are arranged in a line as shown:



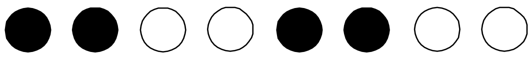
The task is to get all of the black discs to the right-hand side and all of the white discs to the left-hand side. The only move allowed is to interchange two neighbouring discs.

i.e.  becomes 

What is the smallest number of moves that need to be made?

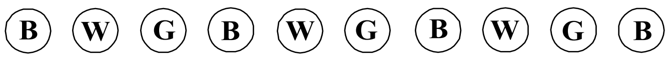
How many moves would it take if we had n black discs and n white discs arranged alternatively?

Suppose the discs are arranged in pairs.



How many moves would it take if there were n each of black and white?

Now suppose that you have three colours, black, white and green.



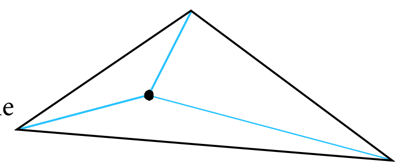
The task here is to get all the black discs to the right, all the green discs to the left and the white discs to the middle. What is the smallest number of moves required if there are n discs of each colour?

10. Prove that $\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta = \dfrac{\sin^2 n\theta}{\sin\theta}$.

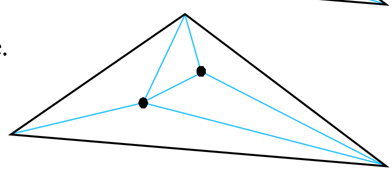
11. Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \dfrac{(2n)!}{(n!)^2}$.

(*Hint:* consider the coefficient of x^n in the expansion of $(1+x)^n(1+x)^n$).

12. Consider placing a point inside a triangle so that non-intersecting lines are drawn from the point to the vertices of the triangle, creating partitions of the larger triangle into smaller triangles. How many partitioned triangles do we have?



Now consider the same problem as above, but this time with two points inside the larger triangle. How many partitioned triangles do we now have?



Make a proposition for this situation when n points are placed inside the triangle. Use the principle of mathematical induction to prove your proposition.

Exercise 1.4.5

1 a $\frac{n(2n^2+3n+7)}{6}$ b $\frac{n^2(n+1)^2}{4}$ c $\frac{1}{4}\left(1-\frac{1}{5^n}\right)$

d $n^2(2n^2-1)$ e $\frac{n(2n^2+9n+7)}{6}$ f $\frac{n}{2n+1}$

2 $\frac{n(n+1)}{2}$

3 $\frac{180(n-2)}{n}$

4 $\frac{n^2+n+2}{2}$

5 $\frac{n(n+1)(2n+1)}{6}$

7 $2n^2-2n+1$

8 $\frac{n^4-6n^3+23n^2-18n+24}{24}$

9 $15, \frac{n(n+1)}{2}, \frac{n(n+2)}{2}, \frac{3n(n+1)}{2}$

12 $2n+1$

Exercise 1.5.1

25. The equation $z + b + i(z - 4) = 0$, where b is a real number, has as its solution a real number. Determine this solution and hence determine the value of b.

26. Express the following in the form $a + bi$ where a and b are real numbers.

a $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos\theta + i\sin\theta}$

b $\dfrac{\cos\theta + i\sin\theta}{\cos 3\theta - i\sin 3\theta}$

27*. Let the complex matrix $A = \begin{bmatrix} \alpha i & 0 \\ 0 & -\beta i \end{bmatrix}$. Find:

a A^2 b A^4

c A^{-1} d A^{4n}, where n is a positive integer.

28*. Find $\dfrac{dy}{d\theta}$ given that $y = \cos\theta + i\sin\theta$.

Show: i $i \cdot \dfrac{d\theta}{dy} = \dfrac{1}{y}$ ii when $\theta = 0, y = 1$.

Hence show that $e^{i\theta} = \cos\theta + i\sin\theta$.

Deduce an expression for $e^{-i\theta}$.

Hence, show: i $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ ii $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.

* questions that are intended for revision. They contain concepts not yet addressed.

Exercise 1.5.1

1 a i 2 ii -3 iii 6 iv 0 v $\frac{3}{2}$ vi $\frac{1}{3}$

b i 2 ii $\sqrt{2}$ iii -5 iv $-\frac{2}{5}$ v $\frac{1}{2}$ vi -1

c i $2-2i$ ii $-3-\sqrt{2}i$ iii $6+5i$

iv $\frac{2}{5}i$ v $\frac{3}{2}-\frac{1}{2}i$ vi $\frac{1}{3}+i$

2 a $7+i$ b $1-3i$ c $15-8i$

d $-1-8i$ e $10+11i$ f $-2+3i$

3 a $-1+3i$ b $5-i$ c $-4+3i$

d $6i$ e $-4+7i$ f $-2+3i$

4 a $\frac{1}{2}(1+i)$ b $-\frac{1}{2}(5+i)$ c $-1-2i$

d $\frac{1}{2}i$ e 1 f $-\frac{1}{13}(5+i)$

5 a $14+8i$ b $-2-2i$ c $-2\sqrt{2}-i$

d $\frac{1}{5}(2+i)$ e $2-i$ f $\frac{1}{5}(1+3i)$

6 a $\frac{1}{2}$ b $\frac{1}{2}(3+\sqrt{2})$ c $3+\sqrt{2}$

7 a $x = 4, y = -\frac{1}{2}$ b $x = -5, y = 12$ c $x = 0, y = 5$

8 a i $1, i, -1, -i, 1, i$ ii $-i, -1, i, 1, -i$

b i -1 ii $-i$ iii -1 iv -1

9 $x = -\frac{120}{29}, y = \frac{39}{29}$

12 a $x = 0$ or $y = 0$ or both b $x^2 - y^2 = 1$

13 a $3-i$ b $2-i$

14 a $4i$ b -4 c $-i$

15 a $x = 13, y = 4$ b $x = 4, y = \frac{4}{3}$

16 1

17 $-\frac{1}{3}(1+2\sqrt{2}i)$

18 $(u, v) = \left(\frac{1}{2}(\sqrt{2}+2), \frac{1}{2}\sqrt{2}\right), \left(\frac{1}{2}(2-\sqrt{2}), -\frac{1}{2}\sqrt{2}\right)$

19 a $-\frac{7}{2}$ b $-\frac{1}{5}$

21 $\pm\frac{\sqrt{2}(1+i)}{2}$

22 a $\cos(\theta+\alpha)+i\sin(\theta+\alpha)$ b $\cos(\theta-\alpha)+i\sin(\theta-\alpha)$

c $r_1r_2(\cos(\theta+\alpha)+i\sin(\theta+\alpha))$

d $x^2-2x\cos(\theta)+1$ e $x^2+2x\sin(\alpha)+1$

24 a $3+i$ b 325 c $(x^2+y^2)^2$

25 $z = 4, b = -4$

26 a $\cos(\theta)+i\sin(\theta)$ b $\cos(4\theta)+i\sin(4\theta)$

27 a $\begin{bmatrix} -\alpha^2 & 0 \\ 0 & -\beta^2 \end{bmatrix}$ b $\begin{bmatrix} \alpha^4 & 0 \\ 0 & \beta^4 \end{bmatrix}$

c $\begin{bmatrix} -\frac{i}{\alpha} & 0 \\ 0 & \frac{i}{\beta} \end{bmatrix}$ d $\begin{bmatrix} \alpha^{4n} & 0 \\ 0 & \beta^{4n} \end{bmatrix}$

28 a $-\sin(\theta)+i\cos(\theta)$ d $\cos(\theta)-i\sin(\theta)$

Exercise 1.6.1

1. Show the following complex numbers on an Argand diagram:

g $\frac{1}{2i}$ h $\frac{2}{1+i}$

3. If $z_1 = 1 + 2i$ and $z_2 = 1 + i$, show each of the following on an Argand diagram:

g $\frac{z_1}{z_2}$ h $\frac{z_2}{z_1}$

4. Find the modulus and argument of:

d $3i$ e $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ f $\frac{1}{\sqrt{2}}(i + 1)$

g 6 h $\left(1 - \frac{1}{2}i\right)^2$

14. Determine the modulus and argument of each of the complex numbers:

a $3 - 4i$ b $\frac{2}{1+i}$ c $\frac{1-i}{1+i}$

15. If $z = 1 + i$ find $Arg(z)$. hence, find $Arg\left(\frac{1}{z^4}\right)$.

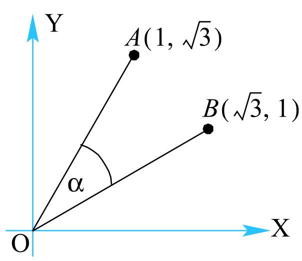
16. Determine the modulus and argument of each of the complex numbers:

a $\cos\theta + i\sin\theta$ b $\sin\theta + i\cos\theta$ c $\cos\theta - i\sin\theta$

17. Find the modulus and argument of:

a $1 + i\tan\alpha$ b $\tan\alpha - i$ c $1 + \cos\theta + i\sin\theta$

18. i Express $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ in the form $u + vi$.



ii Let α be the angle as shown in the diagram. Use part i to find α, clearly explaining your reason(s).

Hence, find $Arg(z)$ where $z = \left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^7$.

19. Find:

i the modulus

ii the principal argument of the complex number $1 - \cos\theta - i\sin\theta$.

On an Argand diagram, for the case $0 < \theta < \pi$, interpret geometrically the relationship:

$$1 - \cos\theta - i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta - \pi}{2}\right) + i\sin\left(\frac{\theta - \pi}{2}\right)\right).$$

20. If $z = \cos\theta + i\sin\theta$, prove:

a $\dfrac{2}{1+z} = 1 - i\tan\left(\dfrac{\theta}{2}\right)$.

b $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$.

Exercise 1.6.1

1 The points to plot are: (2,1), (0,–6), (4,–3), (2,–2), (–3,3),(–3,4),(0.–0.5),(1,–1).

2 a i $i-1$ ii $-(1+i)$ iii $1-i$ iv $1+i$; Anticlockwise rotation of 90°.

b i Reflection about the $Re(z)$ axis.

ii Results will always be a real number, so the point will always lie on $Re(z)$ axis.

iii Point will always lie on the $\mathrm{Im}(z)$ axis.

3 a $-3+4i$ b $\frac{1}{2}(1-i)$ c $-1+3i$ d $1+3i$

e $-1-3i$ f $2-3i$ g $\frac{1}{2}(3+i)$ h $\frac{1}{5}(3-i)$

4 a $2; \frac{\pi}{3}$ b $2; -\frac{\pi}{3}$ c $\sqrt{3}$; $\arctan(\sqrt{2})$ d $3; \frac{\pi}{2}$

e $1; -\frac{2\pi}{3}$ f $1; \frac{\pi}{4}$ g 6; 0 h $\frac{5}{4}; -\arctan\left(\frac{4}{3}\right)$

5 a $\sqrt{a^2+b^2}$; $\sqrt{a^2+b^2}$; a^2+b^2 b i $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ ii 0 or π

6 a $\sqrt{2x^2+18}$ b ± 3

7 Triangle property; the sum of the lengths of two sides of a triangle is larger than the third side.

8 0

9 a 15 b 5 c 10

12 b $1+i$

14 a 5; –53.13° b $\sqrt{2}$; –45° c 1; –90°

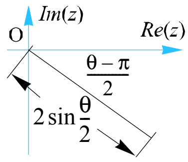
15 $\frac{\pi}{4}, \pi$

16 a $1; \theta$ b $1; \frac{\pi}{2}-\theta$ c $1; -\theta$

17 a $|\sec\alpha|$, $\alpha-\pi$ (for Principal argument) otherwise, $\alpha+k\pi$, where k is an integer.

b $|\sec\alpha|$, $\alpha+\frac{\pi}{2}$ (for Principal argument) otherwise, $\left(a+\frac{p}{2}\right)+kp$, k is an integer.

c $2\left|\cos\left(\frac{\theta}{2}\right)\right|, \frac{\theta}{2}$ (for Principal argument) otherwise, $\frac{\theta}{2}+k\pi$, k is an integer.

18	a i	$\frac{\sqrt{3}}{2}+\frac{i}{2}$	ii	$\frac{\pi}{6}$	b	$-\frac{5\pi}{6}$	
19	a i	$2\sin\frac{\theta}{2}$	ii	$\frac{\theta-\pi}{2}$	b		

Exercise 1.6.2

1	a	$\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$	b	$\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$	c	$\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$	
2	a	$2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$	b	$2\text{cis}\left(\frac{\pi}{6}\right)$	c	$4\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$	
	d	$5\text{cis}(53°7')$	e	$\sqrt{5}\text{cis}(153°26')$	f	$\sqrt{13}\text{cis}(-123°41')$	
	g	$2\text{cis}\left(\frac{5\pi}{6}\right)$	h	$\text{cis}\left(-\frac{\pi}{3}\right)$	i	$\sqrt{10}\text{cis}(-18°26')$	
3	a	$2i$	b	$\frac{3\sqrt{3}}{2}+\frac{3}{2}i$	c	$1-i$	
	d	$-5i$	e	$-4+4\sqrt{3}i$	f	$\frac{1}{6}(\sqrt{2}+\sqrt{6}i)$	
4	a	$\sqrt{\frac{5}{3}}$	b	1	c	0	
5	a	$1-\sqrt{3}i$	b	$1-i$	c	$(1-\sqrt{3})+(1+\sqrt{3})i$	
7	a	$\sqrt{2}$	b	2	c	$2\sqrt{2}$	
	d	$\frac{\pi}{4}$	e	$\frac{2\pi}{3}$	f	$\frac{11\pi}{12}$	

Exercise 1.7.1

9. a If $z = cis(\theta)$, show that:

i $z^2 = \cos(2\theta) + i\sin(2\theta)$

ii $z^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$

Hence, show that:

A $\sin 2\theta = 2\sin\theta\cos\theta$ B $\cos 2\theta = \cos^2\theta - \sin^2\theta$ C $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Using the same approach as that in part a, derive the following identities.

a $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

c $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

10. If $z = cis\theta$, prove: i $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ ii $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

11. If $z = x + iy, y \neq 0$, show that $w = \dfrac{z}{(1+z^2)}$, $1 + z^2 \neq 0$, is real, only if $|z| = 1$.

12. Simplify the expression $\dfrac{1+i\tan\theta}{1-i\tan\theta}$. Hence, show that $\left(\dfrac{1+i\tan\theta}{1-i\tan\theta}\right)^k = \dfrac{1+i\tan k\theta}{1-i\tan k\theta}$ where k is a positive integer.

13. Consider the complex number $z = cis\left(\dfrac{2k\pi}{5}\right)$ for any integer k such that $z \neq 1$.

a Show that $z^n + \dfrac{1}{z^n} = 2\cos\left(\dfrac{2nk\pi}{5}\right)$ for any integer n.

b Show that $z^5 = 1$. Hence, or otherwise, show that $1 + z + z^2 + z^3 + z^4 = 0$.

c Find the value of b, given that $\left(z + \dfrac{1}{z}\right)^2 + \left(z^2 + \dfrac{1}{z^2}\right)^2 = b$.

14. If n is a positive integer, show that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cdot \cos\left(\dfrac{n\pi}{4}\right)$.

15. Simplify the expression $\dfrac{(1 + cis(-\theta))^3}{(1 + cis(\theta))^3}$.

16. If $u = cis\theta$ and $v = cis\alpha$ express $\dfrac{u}{v} + \dfrac{v}{u}$ in terms of θ and α.

17. a If $cis\alpha = a$ and $cis\beta = b$, prove $\sin(\alpha - \beta) = \dfrac{b^2 - a^2}{2ab}i$.

b If $(1 + cis\theta)(1 + cis2\theta) = a + bi$, prove $a^2 + b^2 = 16\cos^2\theta\cos^2\left(\frac{\theta}{2}\right)$.

18. If $|z| = 1$ and $Arg(z) = \theta, 0 < \theta < \frac{\pi}{2}$, find: a $\left|\frac{2}{1 - z^2}\right|$ b $\arg\left(\frac{2}{1 - z^2}\right)$

Exercise 1.7.2

1. Use the *n*th root method to solve the following:

   e $z^4 = 81i$ f $z^6 = -64$

8. If $1, w_1$ and w_2 are the cube roots of unity, prove:

   a $w_1 = \overline{w_2} = w_2^2$ b $w_1 + w_2 = -1$ c $w_1 w_2 = 1$

9. Given that w is a complex root of the equation $z^5 - 1 = 0$ and is such that it has the smallest positive argument, show that w^2, w^3 and w^4 are the other complex roots.

   a Hence show that $1 + w + w^2 + w^3 + w^4 = 0$.

   b Factorise $z^5 - 1$ into real linear and quadratic factors.

   Hence deduce that: i $2\left(\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\right) = -1$

   ii $4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -1$

10. Show that the roots of $(z-1)^6 + (z+1)^6 = 0$ are $\pm i, \pm i\cot\left(\frac{5\pi}{12}\right), \pm i\cot\left(\frac{\pi}{12}\right)$.

Exercise 1.7.1

1 a $-4(1+i)$ b -4 c $-16+16i$
d $-8-8\sqrt{3}i$ e $-16\sqrt{3}-16i$ f $-117-44i$

2 a $\frac{1}{8}(-1+i)$ b $-\frac{1}{4}$ c $-\frac{1}{32}(1+i)$
d $\frac{1}{32}(-1+\sqrt{3}i)$ e $\frac{1}{64}(-\sqrt{3}+i)$ f $\frac{1}{15625}(-117+44i)$

3 a $-8i$ b $\frac{81}{2}(-1+\sqrt{3}i)$ c $\frac{1}{2}i$
d $-\frac{1}{125}i$ e $-\frac{1}{16}(1+\sqrt{3}i)$ f $-\frac{2}{81}(1+\sqrt{3}i)$

4 a $128(1-i)$ b $4\sqrt{3}-4i$ c $-32i$
d 256 e $\frac{11753}{625}-\frac{10296}{625}i$ f $-2i$

5 b $i-1$ ii -1 iii i

6 a $-i$ b $6\sqrt{2}(1+i)$ c $-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}i$

7 a $\frac{\sqrt{2}}{2}(1+i)$; $\frac{1}{2}(1+\sqrt{3}i)$ $\frac{\sqrt{2}}{4}((1-\sqrt{3})+(1+\sqrt{3})i)$
b i $\frac{\sqrt{2}}{4}(1+\sqrt{3})$ ii $\frac{\sqrt{2}}{4}(1-\sqrt{3})$
c $3(\text{cis}(-\theta))^3$

16 $(\cos 2\theta+\cos 2\alpha)(\cos(\theta-\alpha)-i\sin(\theta-\alpha))$ [or $2\cos(\alpha-\theta)$]

18 a $\text{cosec}\,\theta$ b $\theta-\frac{\pi}{2}$

Exercise 1.7.2

1 a $-\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i$, 3 b $\pm\frac{3\sqrt{3}}{2}+\frac{3}{2}i, -3i$ c $2i, \pm\sqrt{3}-i$
d $-\sqrt{2}-\sqrt{2}i, -\sqrt{2}+\sqrt{2}i, \sqrt{2}-\sqrt{2}i, \sqrt{2}+\sqrt{2}i$
e $\frac{3}{2}(\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}i), -\frac{3}{2}(\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}i), \frac{3}{2}(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}i)$ $-\frac{3}{2}(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}i)$
f $\pm 2, \sqrt{3}\pm i, -\sqrt{3}\pm i$

2 $1\pm i, -1\pm i$; $(z-1-i)(z-1+i)(z+1-i)(z+1+i)$

3 a $\pm\frac{1}{\sqrt{2}}(1+i)$ b $2+i, -2-i$ c $\pm\frac{\sqrt{2}}{2}(1+\sqrt{3}i)$

4 a $\sqrt[6]{2}\text{cis}\left(-\frac{\pi}{12}\right), \sqrt[6]{2}\text{cis}\left(\frac{7\pi}{12}\right), \sqrt[6]{2}\text{cis}\left(-\frac{9\pi}{12}\right)$

b $\sqrt[3]{2}\text{cis}\left(\frac{2\pi}{9}\right), \sqrt[3]{2}\text{cis}\left(\frac{8\pi}{9}\right), \sqrt[3]{2}\text{cis}\left(-\frac{4\pi}{9}\right)$

c $\text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(\frac{5\pi}{6}\right), \text{cis}\left(-\frac{\pi}{2}\right)$

5 a $\sqrt[8]{2}\text{cis}\left(\frac{\pi}{16}\right), \sqrt[8]{2}\text{cis}\left(\frac{9\pi}{16}\right), \sqrt[8]{2}\text{cis}\left(-\frac{15\pi}{16}\right), \sqrt[8]{2}\text{cis}\left(-\frac{7\pi}{16}\right)$

b $\text{cis}\left(\frac{\pi}{8}\right), \text{cis}\left(\frac{5\pi}{8}\right), \text{cis}\left(\frac{9\pi}{8}\right), \text{cis}\left(\frac{13\pi}{8}\right)$

c $i, \pm\frac{\sqrt{3}}{2}-\frac{1}{2}i$

d $2\text{cis}\left(-\frac{\pi}{12}\right), 2\text{cis}\left(\frac{5\pi}{12}\right), 2\text{cis}\left(\frac{11\pi}{12}\right), 2\text{cis}\left(-\frac{7\pi}{12}\right)$

e $2(\pm\sqrt{3}+1i)$, $-4i$

f $\pm\frac{1}{2}((\sqrt{3}+1)+(\sqrt{3}-1)i)$

6 a $1, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

7 $-2, 1\pm\sqrt{3}i$

Exercise 1.8.2

11. Given that $2-i$ is a root of $2z^3-9z^2+14z-5 = 0$, find the other roots.

12. Given that $4-i$ is a zero of $P(z) = z^3+az^2+33z-34$, find a and hence factorise $P(z)$.

13. Given that $z-2$ and $z-1-i$ are factors of $P(z) = z^3-az^2+6z+b$, factorise $P(z)$.

14. Solve the following over the real number field.

a $z^6+7z^3-8 = 0$ b $z^6-9z^3+8 = 0$

c $z^4-2z^2-3 = 0$ d $z^4-4z^2-5 = 0$

15. Write down an equation of the lowest possible degree with real coefficients such that its roots are:

a $3, 2-i$ b $2, 1, 1+i$

c $1-\sqrt{3}i, 3$ d $1+\sqrt{2}i, -2+\sqrt{3}i$

16. Verify that $z = -1+\sqrt{3}i$ is a root of the equation $z^4-4z^2-16z-16 = 0$ and hence find the other roots.

17. Given that $z = a+ib$ is a root of $z^4-z^3-6z^2+11z+5 = 0$ and $Re(z) = 2$, solve the equation completely.

18. If $z^n+z^{-n} = 2\cos(n\theta)$ show that $5z^4-z^3-6z^2-z+5 = 0 \Rightarrow 10\cos^2\theta-\cos\theta-8 = 0$.

19. Show that $\cos 5\theta = 16\cos^5\theta-20\cos^3\theta+5\cos\theta$,

and hence show that the roots of $x(16x^4-20x^2+5) = 0$ are 0, $\cos\left(\frac{\pi}{10}\right)$, $\cos\left(\frac{3\pi}{10}\right)$, $\cos\left(\frac{7\pi}{10}\right)$, $\cos\left(\frac{9\pi}{10}\right)$.

Exercise 1.8.1

1 a $(x-3+i)(x-3-i)$ b $(x+2+3i)(x+2-3i)$
c $(x-1+i)(x-1-i)$ d $(z+2+i)(z+2-i)$

e $\left(z-\frac{(3+\sqrt{7}i)}{2}\right)\left(z-\frac{(3+-\sqrt{7}i)}{2}\right)$ f $(z+5+\sqrt{5}i)(z+5-\sqrt{5}i)$

g $4\left(w+\frac{1-4i}{2}\right)\left(w+\frac{1+4i}{2}\right)$ h $3(w-1+i)(w-1-i)$

i $-2\left(w-2-\frac{\sqrt{6}i}{2}\right)\left(w-2+\frac{\sqrt{6}i}{2}\right)$

2 a $-2\pm 2i$ b $\frac{1\pm\sqrt{11}i}{2}$ c $\frac{3\pm\sqrt{3}i}{6}$ d $\frac{-5\pm\sqrt{7}i}{4}$

e $-5\pm 2i$ f $\pm 4i$ g $-6\pm 2i$ h $-3\pm i$
i $\pm\frac{5}{3}i$

3 a $\pm 2, \pm i$ b $\pm 3, \pm i$ c $\pm 3, \pm 2i$

4 a $(z-5i)(z+5i)$ b $(z-7i)(z+7i)$
c $(z+2+i)(z+2-i)$ d $(z+3+\sqrt{2}i)(z+3-\sqrt{2}i)$
e $(z-2i)(z+2i)(z-\sqrt{2})(z+\sqrt{2})$
f $(z-\sqrt{2}i)(z+\sqrt{2}i)(z-\sqrt{3})(z+\sqrt{3})$

Exercise 1.8.2

1 a $(z+2)(z+i)(z-i)$ b $(z-9)(z+i)(z-i)$
c $(z-2)(z+\sqrt{2}i)(z-\sqrt{2}i)$

2 a $(w+1-\sqrt{5}i)(w+1+\sqrt{5}i)(w-2)$
b $(z-1)(z-2+i)(z-2-i)$
c $(z-1)(z+1+i)(z+1-i)$
d $(x+2)(x-2)(x+i)(x-i)$
e $(w+2)(w-1+i)(w-1-i)$
f $(z+5)(z-5)(z+5i)(z-5i)$

3 a $1, 3\pm 4i$ b $2, 3\pm 2i$ c $-2, 3, 1\pm i$
d $\frac{1}{2}, -1\pm i$ e $-\frac{5}{3}, \frac{3}{2}, 1\pm\sqrt{2}i$ f $-1, -3\pm i$

4 $\frac{1}{3}, \frac{-1\pm\sqrt{3}i}{2}$

5 $-\frac{1}{2}, 1\pm 2i$

6 $(z-3)(z-2+3i)(z-2-3i)$

7 $1\pm 2i, \frac{-1\pm\sqrt{11}i}{2}$

8 a $(2z-1)(z+i)(z-i)$ b $(z+\sqrt{3})(z-\sqrt{3})(z+2i)(z-2i)$

9 $2 \pm i, -1$

10 $2 \pm 3i, -\frac{13}{4}$

11 $2 \pm i, \frac{1}{2}$

12 $(z-2)(z-4+i)(z-4-i)$

13 $(z-2)(z-1+i)(z-1-i)$

14 a $-2, 1, \frac{-1 \pm \sqrt{3}i}{2}, 1 \pm \sqrt{3}i$ b $1, 2, \frac{1 \pm \sqrt{3}i}{2}, -1 \pm \sqrt{3}i$

c $\pm\sqrt{3}, \pm i$ d $\pm\sqrt{5}, \pm i$

15 a $x^3 - 7x^2 + 17x - 15 = 0$ b $x^4 - 5x^3 + 10x^2 - 10x + 4 = 0$

c $x^3 - 5x^2 + 10x - 12 = 0$ d $x^4 + 2x^3 + 2x^2 - 2x + 21 = 0$

16 $-1 - i\sqrt{3}, 1 \pm \sqrt{5}$

17 $2 \pm i, \frac{1}{2}(-3 \pm \sqrt{5})$

Exercise 1.9.1

5. Find the solution sets of the following simultaneous equations, solving for x and y.

a
$$\begin{aligned} bx + y &= a \\ ax - y &= b \end{aligned}$$

b
$$\begin{aligned} bx + y &= a \\ ax + y &= b \end{aligned}$$

c
$$\begin{aligned} ax + by &= 1 \\ ax - by &= 1 \end{aligned}$$

d
$$\begin{aligned} ax + y &= ab \\ bx - y &= b^2 \end{aligned}$$

e
$$\begin{aligned} ax + by &= a - b \\ bx + ay &= a - b \end{aligned}$$

f
$$\begin{aligned} ax + y &= b \\ bx + ay &= 2ab - a^3 \end{aligned}$$

Solve Systems of Linear Equations with Matrices (optional)

There is another way to solve a system of linear equation using matrices. The first step is to rewrite the linear equations in standard form. Then you need to identify the coefficient matrix and the resultant matrix. If there is a solution for the system of linear equations, the solution is determined by the product between the inverse of the coefficient matrix and the resultant matrix.

Example 1.9.1

Use matrices to solve: $\begin{aligned} 3x+y-14&=0 \\ -2x+4y&=0 \end{aligned}$

Rewrite the equations in standard form: $\begin{aligned} 3x+y&=14 \\ -2x+4y&=0 \end{aligned}$

The equation can be written in matrix form as: $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

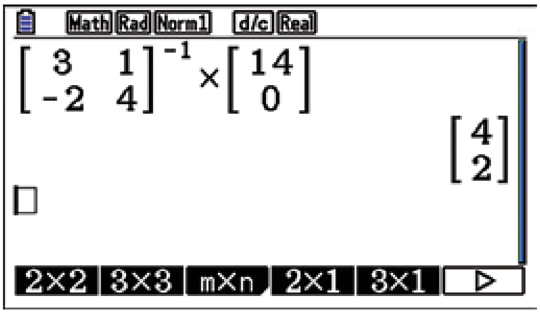
Identify the coefficient matrix: let $A=\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$

Identify the resultant matrix: let $B=\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

Pre-multiplying by the inverse of A: $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

so that: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

The inverse can be found using a calculator:



Therefore, the solution is $x = 4$, $y = 2$.

Exercise 1.9.1

1	a	$x = 1, y = 2$	b	$x = 3, y = 5$	c	$x = -1, y = 2$
	d	$x = 0, y = 1$	e	$x = -2, y = -3$	f	$x = -5, y = 1$
2	a	$x = \frac{13}{11}, y = \frac{17}{11}$	b	$x = \frac{9}{14}, y = \frac{3}{14}$	c	$x = 0, y = 0$
	d	$x = \frac{4}{17}, y = -\frac{22}{17}$	e	$x = -\frac{16}{7}, y = \frac{78}{7}$	f	$x = \frac{5}{42}, y = -\frac{3}{28}$
3	a	-3	b	-5	c	-1.5
4	a	$m = 2, a = 8$	b	$m = 10, a = 24$	c	$m = -6, a = 9.$
5	a	$x = 1, y = a - b$	b	$x = -1, y = a + b$	c	$x = \frac{1}{a}, y = 0$
	d	$x = b, y = 0$	e	$x = \frac{a-b}{a+b}, y = \frac{a-b}{a+b}$	f	$x = a, y = b - a^2$

Exercise 1.9.2

1	a	$x = 4, y = -5, z = 1$	b	$x = 0, y = 4, z = -2$
	c	$x = 10, y = -7, z = 2$	d	$x = 1, y = 2, z = -2$
	e	$\varnothing$	f	$x = 2t - 1, y = t, z = t$
	g	$x = 2, y = -1, z = 0$	h	$\varnothing$

Exercise 2.1.1

2 Find the range for each of the following.

i $y = \sqrt{x}, x \geq 0$ j $y = \sqrt{x}, 1 \leq x \leq 25$

k $y = \frac{4}{x+1}, x > 0$ l $\{(x,y): y^2 = x, x \geq 1\}$

4 Determine the implied domain for each of the following relations.

h $y = \sqrt{x+a}, a > 0$ i $y = \frac{a}{\sqrt{x-a}}, a > 0$

j $x^2 - y^2 = a^2$ k $y^2 - x^2 = a^2$

5 Find the range of the following relations.

f $y = a - \frac{a}{x^2}, a > 0$

g $y = 2\sqrt{x-a} - a, a > 0$ h $y = \frac{2a}{\sqrt{a^2 - x}}, a < 0$

Exercise 2.1.2

5. The function f is defined as $f:]-\infty, \infty[\mapsto \mathbb{R}$, where $f(x) = x^2 - 4$.

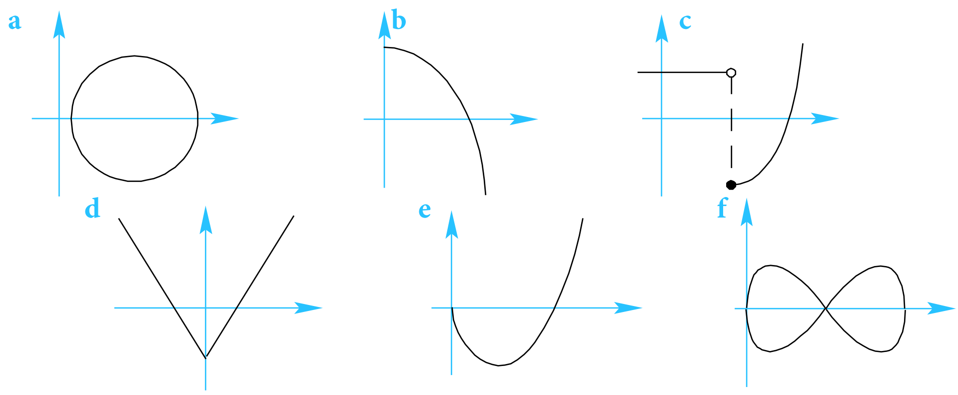
a Sketch the graph of:

i f ii $y = x + 2, x \in]-\infty, \infty[$

b Find:

i $\{x : f(x) = 4\}$ ii $\{x : f(x) = x + 2\}$

6 Which of the following relations are also functions?



7 Use both visual and algebraic tests to show that the following relations are also functions:

a $x \mapsto x^3 + 2, x \in]0,5[$ b $x \mapsto \sqrt{x} + 1, x \in [0, 9[$

c $\{(x,y) : y^3 = x + 1, x \in \mathbb{R}\}$ d $\{(x,y) : y = x^2 + 1, x \in \mathbb{R}\}$

8 Use an algebraic method to decide which of the following relations are also functions:

a $f : x \mapsto \frac{1}{x}, x \in \mathbb{R}\backslash\{0\}$ b $\{(x,y) : y^2 - x = 9, \ x \geq -9\}$

c $\{(x,y) : y^2 - x^2 = 9, \ x \geq -9\}$ d $f(x) = \frac{1}{x^2} + 1, x \neq 0$

e $f(x) = 4 - 2x^2, x \in \mathbb{R}$ f $f : x \mapsto \frac{4}{x+1}, x \in \mathbb{R}\backslash\{-1\}$

9 Sketch the graph of $f : \mapsto \frac{x^2}{x^2+2}, x \in \mathbb{R}$ and use it to:

a show that f is a function b determine its range.

10 A function is defined by $f : x \mapsto \frac{x+10}{x-8}, x \neq 8$ and $x \geq 0$.

a Determine the range of f.

b Find the value of a such that $f(a) = a$.

11 Consider the functions $h(x) = \frac{1}{2}(2^x + 2^{-x})$ and $k(x) = \frac{1}{2}(2^x - 2^{-x})$.

a Show that $2[h(x)]^2 = h(2x) + 1$.

b If $[h(x)]^2 - [k(x)]^2 = a$, find the constant a.

12 Which of the following functions are identical? Explain.

a $f(x) = \frac{x}{x^2}$ and $h(x) = \frac{1}{x}$. b $f(x) = \frac{x^2}{x}$ and $h(x) = x$.

c $f(x) = x$ and $h(x) = \sqrt{x^2}$ d $f(x) = x$ and $h(x) = (\sqrt{x})^2$.

13 Find the largest possible subset X of $\mathbb{R}$, so that the following relations are one-to-one increasing functions:

a $f: X \to \mathbb{R}$, where $f(x) = x^2 + 6x + 10$

b $f: X \to \mathbb{R}$, where $f(x) = \sqrt{9 - x^2}$

c $f: X \to \mathbb{R}$, where $f(x) = \sqrt{x^2 - 9}$

d $f: X \to \mathbb{R}$, where $f(x) = \dfrac{1}{3x - x^2}$, $x \neq 0, 3$

14 An isosceles triangle ABC has two side lengths measuring 4 cm and a variable altitude. Let the altitude be denoted by x cm.

a Find, in terms of *x*, a relation for:

i its perimeter, $p(x)$ cm and specify its implied domain.

ii its area, $A(x)$ cm^2 and specify its implied domain.

b Sketch the graph of:

i $p(x)$ and determine its range.

ii $A(x)$ and determine its range.

Exercise 2.1.4

3 All of the following functions are mappings of $\mathbb{R}\mapsto\mathbb{R}$ unless otherwise stated.

a Determine the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$, if they exist.

b For the composite functions in part a that do exist, find their range.

viii $f(x) = x - 4,\ g(x) = |x|$

ix $f(x) = x^3 - 2,\ g(x) = |x+2|$

x $f(x) = \sqrt{4-x},\ x \le 4,\ g(x) = x^2$

xi $f(x) = \dfrac{x}{x+1},\ x \ne -1,\ g(x) = x^2$

xii $f(x) = x^2 + x + 1,\ g(x) = |x|$

xiii $f(x) = 2^x,\ g(x) = x^2$

xiv $f(x) = \dfrac{1}{x+1},\ x \ne -1,\ g(x) = x - 1$

xv $f(x) = \dfrac{2}{\sqrt{x-1}},\ x > 1,\ g(x) = x^2 + 1$

xvi $f(x) = 4^x,\ g(x) = \sqrt{x}$

13 Find $(h \circ f)(x)$, given that $h(x) = \begin{cases} x^2 + 4, & x \ge 1 \\ 4 - x, & x < 1 \end{cases}$ and $f : x \mapsto x - 1,\ x \in \mathbb{R}$.

Sketch the graph of $(h \circ f)(x)$ and use it to find its range.

14 a Given three functions, f, g and h, when would $h \circ g \circ f$ exist?

b If $f : x \mapsto x + 1,\ x \in \mathbb{R}$, $g : x \mapsto x^2,\ x \in \mathbb{R}$ and $h : x \mapsto 4x,\ x \in \mathbb{R}$, find $(h \circ g \circ f)(x)$.

15 Given the functions $f(x) = e^{2x-1}$ and $g(x) = \frac{1}{2}(\ln x + 1)$ find, where they exist:

a $(f \circ g)$ b $(g \circ f)$ c $(f \circ f)$

In each case find the range of the composite function.

16 Given that $h(x) = \log_{10}(4x - 1),\ x > \frac{1}{4}$ and $k(x) = 4x - 1,\ x \in\ ]-\infty,\infty[$, find, where they exist :

a $(h \circ k)$ b $(k \circ h)$.

17 Given the functions $f(x) = \sqrt{x^2 - 9},\ x \in \mathbf{S}$ and $g(x) = |x| - 3,\ x \in \mathbf{T}$, find the largest positive subsets of $\mathbb{R}$ so that:

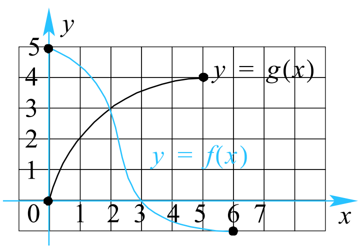
a $g \circ f$ exists b $f \circ g$ exists.

18 For each of the following functions:

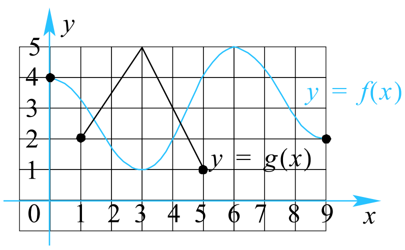
a determine if $f \circ g$ exists and sketch the graph of $f \circ g$ when it exists.

b determine if $g \circ f$ exists and sketch the graph of $g \circ f$ when it exists.

i



ii



19 Given the functions $f : S \to \mathbb{R}$ where $f(x) = e^{x+1}$ and $g : S \to \mathbb{R}$ where $g(x) = \ln 2x$ where $S =]0, \infty[$.

a State the domain and range of both f and g.

b Giving reasons, show that $g \circ f$ exists but $f \circ g$ does not exist.

c Fully define $g \circ f$, sketch its graph and state its range.

20 The functions f and g are given by $f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geq 1 \\ x-1 & \text{if } 0 < x < 1 \end{cases}$ and $g(x) = x^2 + 1$.

a Show that $f \circ g$ is defined. b Find $(f \circ g)(x)$ and determine its range.

21 Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ where $f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \leq 1 \\ \dfrac{1}{\sqrt{x}}, & x > 1 \end{cases}$.

a Sketch the graph of f.

b Define the composition $f \circ f$, justifying its existence.

c Sketch the graph of $f \circ f$, giving its range.

22 Consider the functions $f :]1,\infty[\rightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$ and $g : \mathbb{R} \backslash \{0\} \rightarrow \mathbb{R}$ where $g(x) = x^2$.

a Sketch the graphs of f and g on the same set of axes.

b Prove that $g \circ f$ exists and find its rule.

c Prove that $f \circ g$ cannot exist.

d If a new function $g^* : S \rightarrow \mathbb{R}$ where $g^*(x) = g(x)$ is now defined, find the largest positive subset of $\mathbb{R}$ so that $f \circ g^*$ does exist. Find $f \circ g^*$, sketch its graph and determine its range.

23 Given that $f(x) = \dfrac{ax-b}{cx-a}$, show that $f \circ f$ exists and find its rule.

24 a Sketch the graphs of $f(x) = \dfrac{1}{a}x^2$ and $g(x) = \sqrt{2a^2 - x^2}$, where $a > 0$.

b Show that $f \circ g$ exists, find its rule and state its domain.

c Let S be the largest subset of $\mathbb{R}$ so that $g \circ f$ exists.

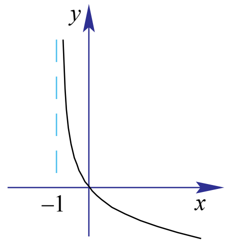
i Find S.

ii Fully define $g \circ f$, sketch its graph and find its range.

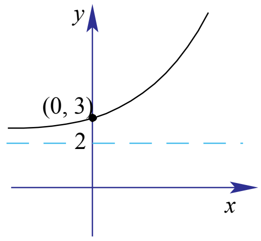
Exercise 2.1.5

5 Sketch the inverse of the following functions.

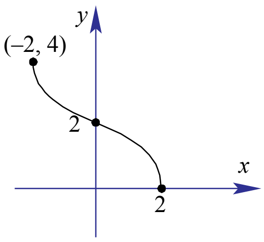
e



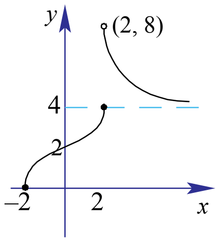
f



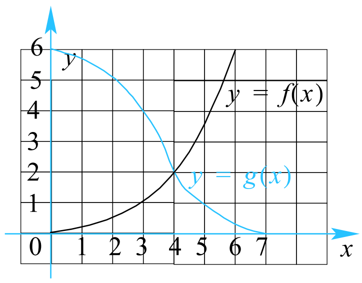
g



h



18 Consider the functions f and g:



a Does $g \circ f$ exist? Justify your answer.

b Does $(g \circ f)^{-1}$ exist? Justify your answer.

If it does exist, sketch the graph of $(g \circ f)^{-1}$.

19 a On the same set of axes, sketch the graph of $f(x) = -x^3$ and its inverse, $f^{-1}(x)$.

b The function g is given by $g(x) = \begin{cases} 2x+1, & x < -1 \\ -x^3, & -1 \le x \le 1 \\ 2x-1, & x > 1 \end{cases}$.

i Sketch the graph of g.

ii Fully define its inverse, g^{-1}, stating why it exists.

iii Sketch the graph of g^{-1}.

iv Find $\{x : g(x) = g^{-1}(x)\}$.

20 Consider the functions $t(x) = e^x$ and $m(x) = \sqrt{x}$.

a Find, where they exist, the composite functions: **i** $(t \circ m)(x)$ **ii.** $(m \circ t)(x)$

b With justification, find and sketch the graphs of: **i** $(t \circ m)^{-1}(x)$ **ii** $(m \circ t)^{-1}(x)$

c Find: **i** $t^{-1} \circ m^{-1}(x)$ **ii** $m^{-1} \circ t^{-1}(x)$

d What conclusion(s) can you make from your results of parts **b** and **c**?

e Will your results of part **d** work for any two functions f and g? Explain.

21 **a** Find $\{x : x^3 + x - 2 = 0\}$.

b If $f(x) = \frac{1}{\sqrt{x}} - 2$, sketch the graph of $y = f(x)$ and find $\{x : f(x) = f^{-1}(x)\}$.

22 Consider the functions $f(x) = |x|, x \in \mathbf{A}$ and $g(x) = e^x - 2, x \in \mathbf{B}$.

a Sketch the graphs of: **i** f if $\mathbf{A} = \mathbb{R}$ **ii** g if $\mathbf{B} = \mathbb{R}$.

b With **A** and **B** as given in part **a**, give reasons why $(f \circ g)^{-1}$ will not exist.

c **i** Find the largest set **B** which includes positive values, so that $(f \circ g)^{-1}$ exists.

ii Fully define $(f \circ g)^{-1}$.

iii On the same set of axes, sketch the graphs of $(f \circ g)(x)$ and $(f \circ g)^{-1}(x)$.

Exercise 2.1.1

1 a dom = {2, 3, −2}, ran = {4, −9, 9}
b dom = {1, 2, 3, 5, 7, 9}, ran = {2, 3, 4, 6, 8, 10}
c dom = {0, 1}, ran = {1, 2}

2 a]1, ∞[b [0, ∞[c]9, ∞[
d]−∞, 1] e [−3, 3] f]−∞, ∞[
g]−1, 0] h [0, 4] i [0, ∞[
j [1, 5] k]0, 4[l]−∞, −1] ∪ [1, ∞[

3 a r = [−1, ∞[, d = [0, 2[b $r = \{y : y \geq 0\} \backslash \{4\}$, $d = \mathbb{R}$
c r = [0, ∞[\{3}, d = [−4, ∞[\{0} d r = [−2, 0[, d = [−1, 2[
e r =]−∞, ∞[d =]−∞, −3] ∪ [3, ∞[f r = [−4,4], d = [0,8]

4 a $\mathbb{R}\backslash\{-2\}$ b]−∞, 9[c [−4,4]
d]−∞, −2] ∪ [2, ∞[e $\mathbb{R}\backslash\{0\}$ f $\mathbb{R}$
g $\mathbb{R}\backslash\{-1\}$ h [−a, ∞[i [0, ∞[\{a^2}
j]−∞, −a] ∪ [a, ∞[k $\mathbb{R}$ l $\mathbb{R}\backslash\{-a^{-1}\}$

5 a]−∞,−a[b]0,ab] c $]-\infty, \frac{1}{4}a^3]$ d $[\frac{1}{4}a^3, \infty[$
e $\mathbb{R}\backslash\{a\}$ f]−∞,a[g [−a,∞[h]−∞, 0[

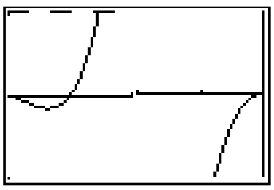
Exercise 2.1.2

1 a 3, 5 b i 2(x+a) + 3 ii 2a c 3

2 a $0, \frac{10}{11}$ b $-\frac{5}{4}$ c $\left[0, \frac{10}{11}\right]$

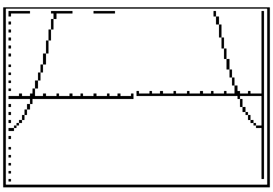
3 a $-\frac{1}{2}x^2 - x + \frac{3}{2}$, $-\frac{1}{2}x^2 + x + \frac{3}{2}$ b $\pm\sqrt{2}$ c no solution

4 a x = 0, 1 **b**

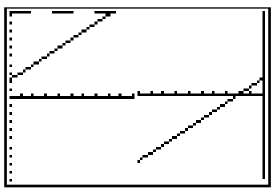


Window [−2,2], [−1,1]
Range: [−12, 4]

5 ai



ii

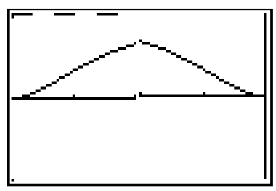


b i $\{2\sqrt{2}, -2\sqrt{2}\}$ ii {3, −2}

6 b, c, d, e

8 a, d, e, f

9 a Window [−2,2], [−1,1] **b** [0, 1[



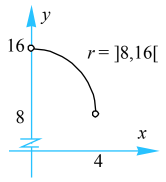
10 a $\{y: y > 1\} \cup \{y: y \leq -1.25\}$
b 10

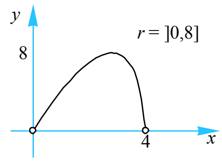
11 b 1

12 a only – it is the only one with identical rules and domains

13 a $[-3,\infty[$ b $[-3,0]$ c $[3,\infty[$
d $[1.5,3[\ \cup\ ]3,\infty[$

14 a i $p(x) = 8 + 2\sqrt{16 - x^2},\ 0 < x < 4$ ii $A(x) = x\sqrt{16 - x^2},\ 0 < x < 4$

b i 

ii 

Exercise 2.1.3

1 a even b even c neither d neither
e even f odd g odd h even
i odd

3 Not if 0 is excluded from the domain.

6 $f(x) = 0,\ x \in \Re$

Exercise 2.1.4

1 a i $f + g\colon [0, \infty[\mapsto \mathbb{R}$ where $(f+g)(x) = x^2 + \sqrt{x}$ $[0,\infty[$

ii $f + g\colon]0, \infty[\mapsto \mathbb{R}$ where $(f+g)(x) = \frac{1}{x} + \ln(x)$ $[1,\infty[$

iii $f + g\colon [-3, -2] \cup [2, 3] \mapsto \mathbb{R}$ where $(f+g)(x) = \sqrt{9 - x^2} + \sqrt{x^2 - 4}$, $[\sqrt{5}, \sqrt{10}]$

b i $fg\colon [0, \infty[\mapsto \mathbb{R}$ where $(fg)(x) = x^2\sqrt{x} = x^{5/2}$

ii $fg\colon]0, \infty[\mapsto \mathbb{R}$ where $(fg)(x) = \frac{\ln(x)}{x}$

iii $fg\colon [-3, -2] \cup [2, 3] \mapsto \mathbb{R}$ where $(fg)(x) = \sqrt{(9 - x^2)(x^2 - 4)}$

2 a i $f–g\colon]-\infty, \infty[\mapsto \mathbb{R}$ where $(f–g)(x) = 2e^x - 1$ $]-1,\infty[$

ii $f–g\colon]-1, \infty[\mapsto \mathbb{R}$ where $(f–g)(x) = (x+1) - \sqrt{x+1}$ $]-0.25,\infty[$

iii $f–g\colon]-\infty, \infty[\mapsto \mathbb{R}$ where $(f–g)(x) = |x-2| - |x+2|$, $[-4,4]$

b i $f/g\colon \mathbb{R}\backslash\{0\}, \mapsto \mathbb{R}$ where $(f/g)(x) = \frac{e^x}{1 - e^x}$

ii $f/g\colon]-1, \infty[\mapsto \mathbb{R}$ where $(f/g)(x) = \sqrt{x+1}$

iii $f/g\colon \mathbb{R}\backslash\{-2\} \mapsto \mathbb{R}$ where $(f/g)(x) = \left|\frac{x-2}{x+2}\right|$

3 a $fog(x) = x^3 + 1$, $gof(x) = (x+1)^3$ b $]-\infty, \infty[$, $]-\infty, \infty[$

ii a $fog(x) = x + 1, x \geq 0$, $gof(x) = \sqrt{x^2 + 1}$ b $[1, \infty[$, $[1, \infty[$

iii a $fog(x) = x^2$, $gof(x) = (x+2)^2 - 2$ b $[0, \infty[$, $[-2, \infty[$

iv a $fog(x) = x, x \neq 0$, $gof(x) = x, x \neq 0$ b $\mathbb{R}\backslash\{0\}$, $\mathbb{R}\backslash\{0\}$

v a $fog(x) = x, x \geq 0$, $gof(x) = |x|$ b $[0, \infty[$, $[0, \infty[$

vi a $fog(x) = \frac{1}{x^2} - 1, x \neq 0$, $gof(x)$ does not exist. b $]-1, \infty[$

vii a $fog(x) = x^2, x \neq 0$, $gof(x) = x^2, x \neq 0$ b $]0, \infty[$, $]0, \infty[$

viii a $fog(x) = |x| - 4$, $gof(x) = |x - 4|$ b $[-4, \infty[$, $[0, \infty[$

ix a $fog(x) = |x+2|^3 - 2$, $gof(x) = |x^3|$ b $[-2, \infty[$, $[0, \infty[$

x a $fog(x)$ does not exist, $gof(x) = (4 - x), x \leq 4$ b $[0, \infty[$

xi a $fog(x) = \frac{x^2}{x^2+1}$, $gof(x) = \left(\frac{x}{x+1}\right)^2, x \neq -1$ b $[0,1[$, $[0, \infty[$

xii a $fog(x) = x^2 + |x| + 1$, $gof(x) = |x^2 + x + 1|$ b $[1, \infty[$, $[0.75, \infty[$

xiii a $fog(x) = 2^{x^2}$, $gof(x) = 2^{2x}$ b $[1, \infty[$, $]0, \infty[$

xiv a $fog(x)$ does not exist, $gof(x) = \frac{1}{x+1} - 1, x \neq -1$ b $\mathbb{R}\backslash\{-1\}$

xv a $fog(x)$ does not exist, $gof(x) = \frac{4}{x-1} + 1$ b $]1, \infty[$

xvi a $fog(x) = 4^{\sqrt{x}}, x \geq 0$, $gof(x) = 4^{0.5x}$ b $[1, \infty[$, $]0, \infty[$

4 a $fog(x) = 2x + 3, x \in \mathbb{R}$
b $gof(x) = 2x + 2, x \in \mathbb{R}$
c $fof(x) = 4x + 3, x \in \mathbb{R}$

5 $g(x) = x^2 + 1, x \in \mathbb{R}$

6 a $fog(x) = \frac{1}{x} + x + 1, x \in \mathbb{R}\backslash\{0\}$, $]-\infty,-1] \cup [3,\infty[$
b $gof(x)$ does not exist.

c $gog(x) = x + \frac{1}{x} + \frac{x}{x^2+1}, x \neq 0$, $]-\infty,-2.5] \cup [2.5,\infty[$

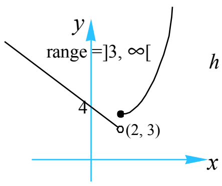
7 a 9 b 3

9 $g(x) = x^2 + 3$

10 $g(x)=\frac{1}{2}\sqrt{x^2-1}+2$

11 a $x = \pm 1$ b $x = 1, -3$

12 a $\frac{1}{x}$ b $\frac{-x}{2x+1}$

13 

$hof(x) = \begin{cases} (x-1)^2+4, & x \geq 2 \\ 5-x, & x < 2 \end{cases}$

14 a $r_f \subseteq d_g$ and $r_{gof} \subseteq d_h$ b $g(x) = 4(x+1)^2, x \in \mathbb{R}$

15 a $fog(x) = x, x \in]0,\infty[$ range $=]0,\infty[$

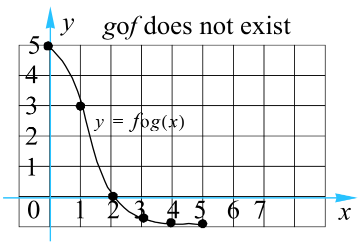
b $gof(x) = \frac{1}{2}(\ln(e^{2x-1})+1), x \in \mathbb{R}\ (= x)$ range $=]-\infty,\infty[$

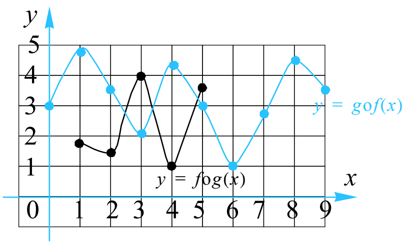
c $fof(x) = e^{2(e^{2x-1})-1}, x \in \mathbb{R}$ range $=]e^{-1}$,¥[

16 a hok does not exist. b $koh(x) = 4\log(4x-1)-1, x > \frac{1}{4}, \mathbb{R}$

17 a $S = \mathbb{R}\backslash]-3,3[$; $T = \mathbb{R}$

b $T = \{x : |x| \geq 6, x = 0\}$; $S =]-\infty,-3] \cup [3,\infty[$

18 

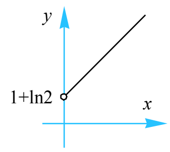


19 a Dom $f =]0,\infty[$, ran $f =]e,\infty[$, Dom $g =]0,\infty[$, ran $g = \mathbb{R}$

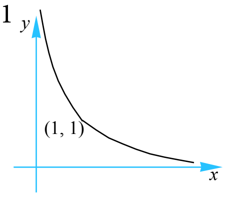
b fog does not exist: $r_g = \mathbb{R} \not\subset d_f =]0,\infty[$

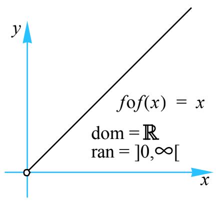
gof exists as $r_f =]e, \infty[\subseteq d_g =]0,\infty[$

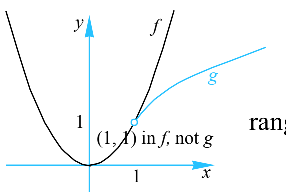
c gof: $]0,\infty[\mapsto \mathbb{R}$, where $gof(x) = (x+1)+\ln 2$



20 $(fog)(x) = |x|, x \in \mathbb{R}$; range $= [0, \infty[$

21 

c 

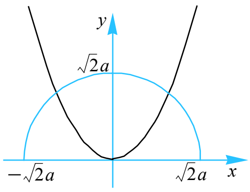
22 

range $=]1,\infty[$

b gof: $]1,\infty[\mapsto \mathbb{R}$, where $gof(x) = x$

d fog^*: $]1,\infty[\mapsto \mathbb{R}$, where $gof(x) = x$

$d_f = \mathbb{R}\backslash\left\{\frac{a}{c}\right\}, r_f = \mathbb{R}\backslash\left\{\frac{a}{c}\right\}, r_f \subseteq d_f, fof(x) = x$

24 a 

b $d_{fog} = [-\sqrt{2}a, \sqrt{2}a], fog = 2a - \frac{x^2}{a}$

c $d_{gof} = [-2^{1/4}a, 2^{1/4}a], fog = \frac{1}{a}\sqrt{2a^4 - x^4}$,

$range = [0, \sqrt{2}a]$

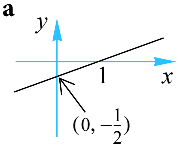
Exercise 2.1.5

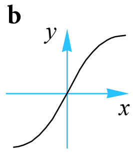
1 a $\frac{1}{2}(x-1), x \in \mathbb{R}$ b $\sqrt[3]{x}, x \in \mathbb{R}$

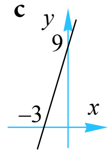
c $3(x+3), x \in \mathbb{R}$ d $\frac{5}{2}(x-2), x \in \mathbb{R}$

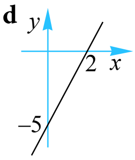
e $x^2 - 1, x > 0$ f $(x-1)^2, x \geq 1$

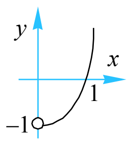
g $\frac{1}{x} - 1, x > 0$ h $\frac{1}{(x+1)^2}, x > -1$

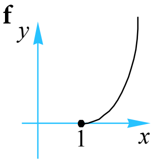
2 a 

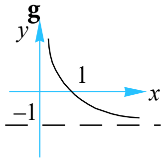
b 

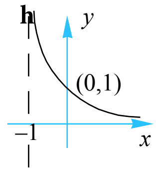
c 

d 

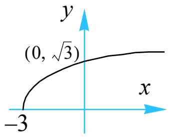
e 

f 

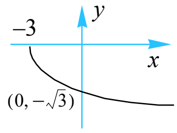
g 

h 

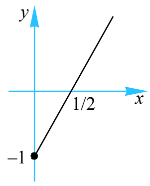
3 a $\sqrt{x+3}, x \geq -3$

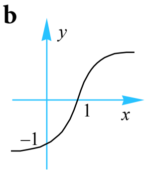


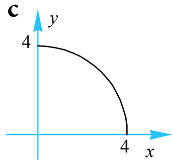
b $-\sqrt{x+3}, x \geq -3$

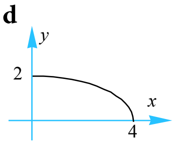


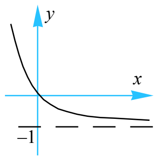
4 $\frac{\pm|x|}{\sqrt{1-x^2}}, -1 < x < 1$

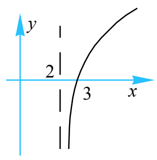
5 a 

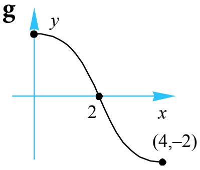
b 

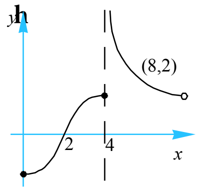
c 

d 

e 

f 

g 

h 

6

a $f^{-1}(x) = \log_3(x-1), x > 1$

b $f^{-1}(x) = \log_2(x+5), x > -5$

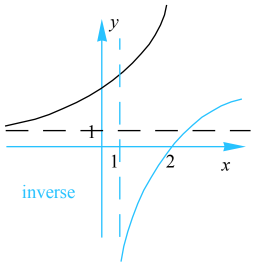
c $f^{-1}(x) = \frac{1}{2}(\log_3 x - 1), x > 0$

d $g^{-1}(x) = 1 + \log_{10}(3-x), x < 3$

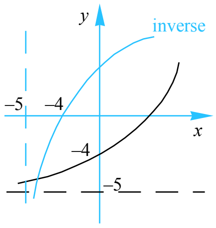
e $h^{-1}(x) = \log_3\left(1 + \frac{2}{x}\right), x \in \mathbb{R}\backslash[-2,0]$

f $g^{-1}(x) = \log_2\left(\frac{1}{x+1}\right), x > -1$

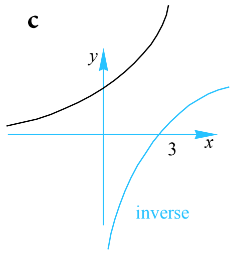
7 a



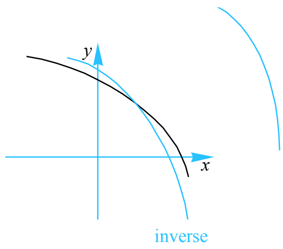
b



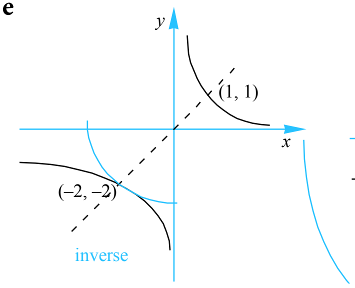
c



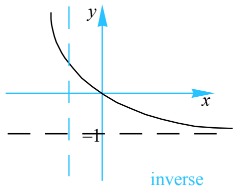
d



e



f



8 a $f^{-1}(x) = 2^x - 1, x \in \mathbb{R}$

b $f^{-1}(x) = \frac{1}{2} \cdot 10^x, x \in \mathbb{R}$

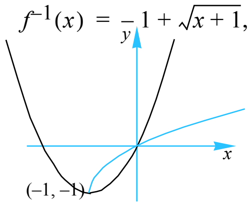
c $f^{-1}(x) = 2^{1-x}, x \in \mathbb{R}$

d $f^{-1}(x) = 3^{x+1} + 1, x \in \mathbb{R}$

e $f^{-1}(x) = 5^{x/2} + 5, x \in \mathbb{R}$

f $f^{-1}(x) = 1 - 10^{3(2-x)}, x \in \mathbb{R}$

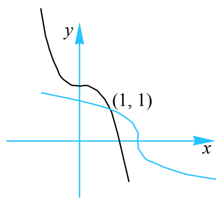
9 $f^{-1}(x) = -1 + \sqrt{x+1}, x > -1$



dom = [–1, ∞[, ran = [–1, ∞[

10 a $f^{-1}(x) = a - x$ **b** $f^{-1}(x) = \frac{2}{x-a} + a$ **c** $f^{-1}(x) = \sqrt{a^2 - x^2}$

11 $f^{-1}(x) = \sqrt[3]{2-x}$



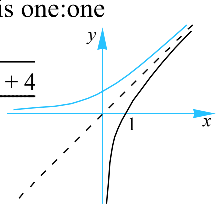
12 [2, ∞[

13 $\mathbb{R}^+ \backslash \{1.5\}$

14 a Inverse exists as *f* is one:one

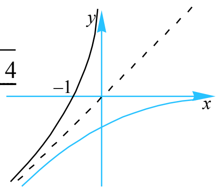
b Case 1: $S =]0, \infty[$

$g^{-1}(x) = \frac{x + \sqrt{x^2 + 4}}{2}$

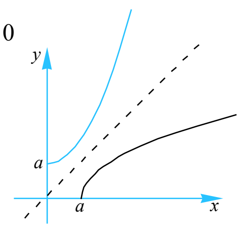


Case 2: $S =]-\infty, 0[$

$g^{-1}(x) = \frac{x - \sqrt{x^2 + 4}}{2}$

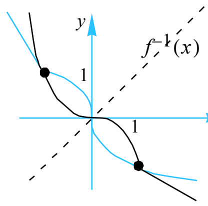


15 $f^{-1}(x) = a(x^2 + 1), x \geq 0$



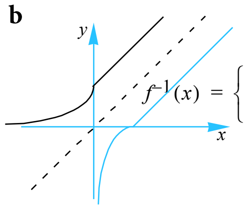
$\{x: f(x) = f^{-1}(x)\} = \varnothing$

16 a



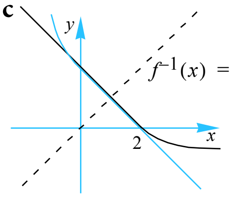
$f^{-1}(x) = \begin{cases} -(2x+1), & x < -1 \\ -\sqrt[3]{x}, & x \geq -1 \end{cases}$

b



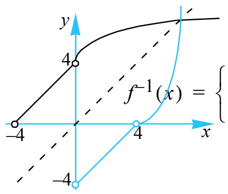
$f^{-1}(x) = \begin{cases} \ln(x) - 1, & 0 < x \leq e \\ x - e, & x > e \end{cases}$

c



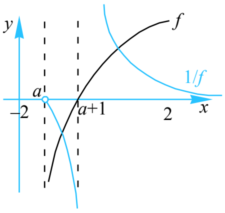
$f^{-1}(x) = \begin{cases} 1 + e^{-x}, & x < 0 \\ 2 - x, & x \geq 0 \end{cases}$

d

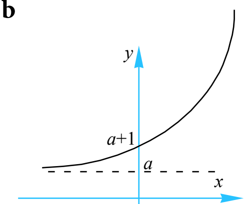


$f^{-1}(x) = \begin{cases} (x-4)^2, & x > 4 \\ x - 4, & 0 < x < 4 \end{cases}$

17 a



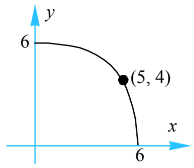
b



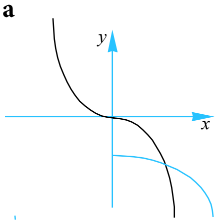
$f^{-1}: \mathbb{R} \mapsto \mathbb{R}, f^{-1}(x) = a + e^{ax}$

18 $g o f$ exists as $r_f \subseteq d_g$.

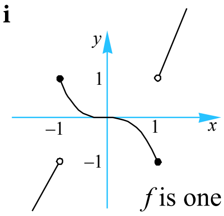
It is one:one so the inverse exists:



19 a



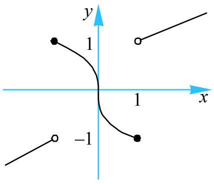
b i



f is one:one

ii $f(x) = \begin{cases} \frac{1}{2}(x-1) & x < -1 \\ -\sqrt[3]{x} & -1 \leq x \leq 1 \\ \frac{1}{2}(x+1) & x > 1 \end{cases}$

iii



iv $\{-1, 0, 1\}$

20 a i $tom(x) = e^{\sqrt{x}}, x \geq 0$ ii $mot(x) = \sqrt{e^x}, x \in \mathbb{R}$

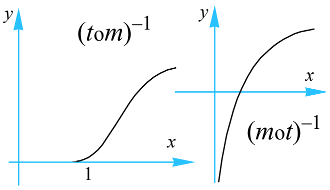
b i $(tom)^{-1}(x) = (\ln(x))^2, x > 1$ ii $(mot)^{-1}(x) = \ln x^2, x > 0$

c i & ii neither exist

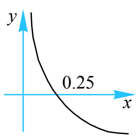
d Adjusting domains so that the functions in part c exist, we have:

$t^{-1}om^{-1}(x) = (mot)^{-1}(x)$ and $m^{-1}ot^{-1}(x) = (tom)^{-1}(x)$

e Yes as rules of composition OK.

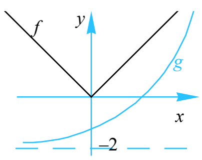


21 a 1 b



0.206

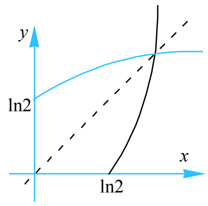
22 a



b fog exists but is not one:one

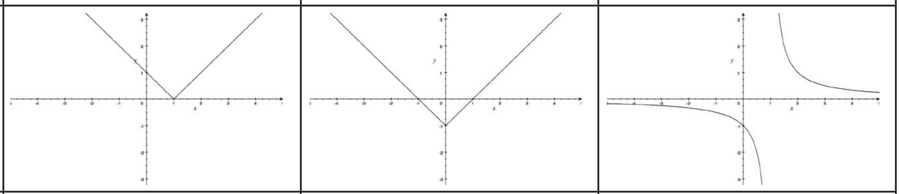
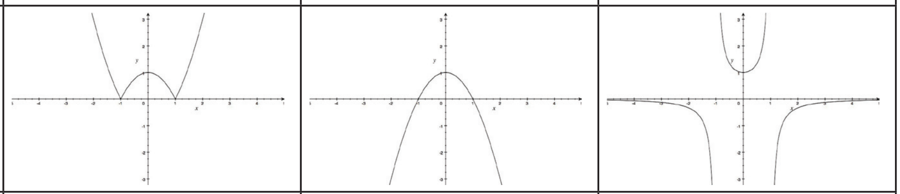
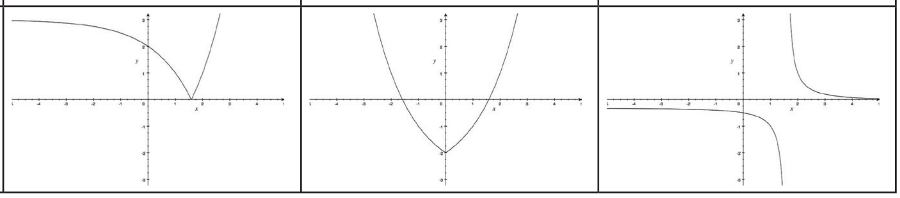
c i B = [ln2, ∞[

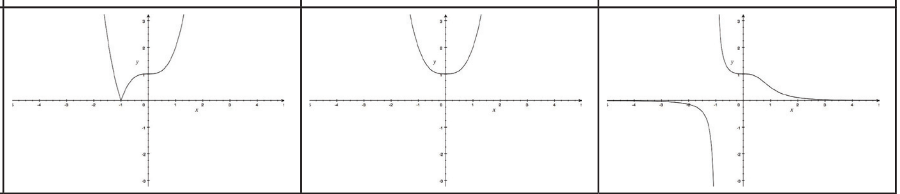
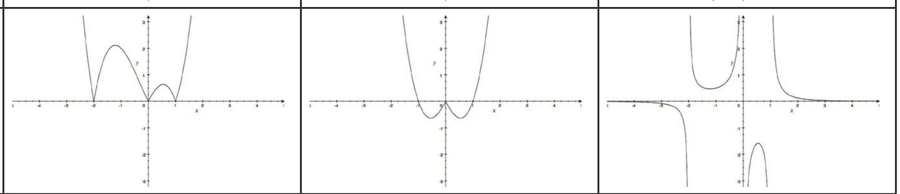
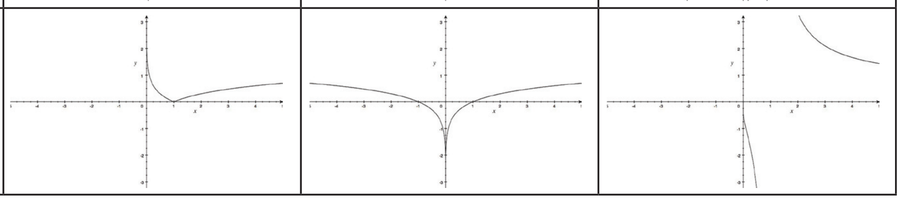
ii $(f\circ g)^{-1}$: [0,∞[↦ ℝ where, $(f\circ g)^{-1}(x) = \ln(x+2)$ iii



Exercise 2.2.1

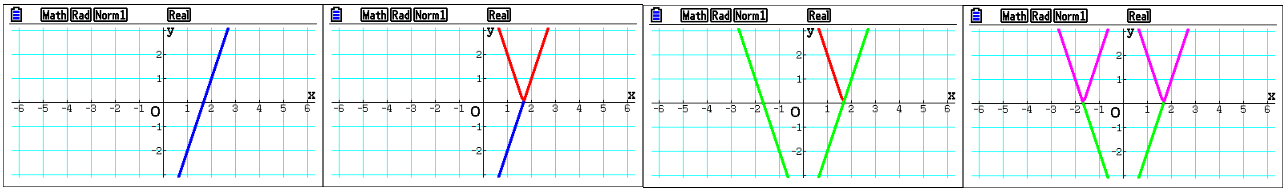
1

	i $y=\lvert f(x)\rvert$	ii $y=f(\lvert x\rvert)$	iii $y=\dfrac{1}{f(x)}$
1			
2			
3			

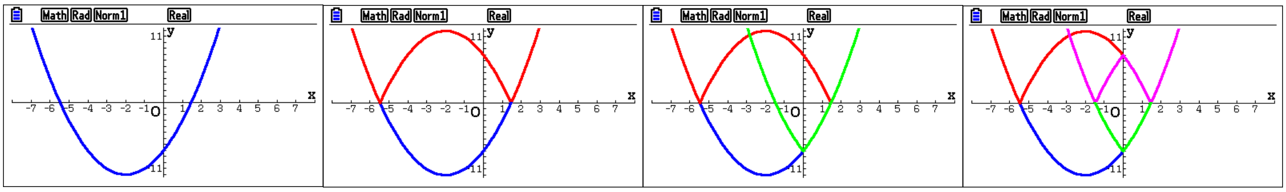
	i $y=\lvert f(x)\rvert$	ii $y=f(\lvert x\rvert)$	iii $y=\dfrac{1}{f(x)}$
4			
5			
6			

2. Adding graphs in the order given:

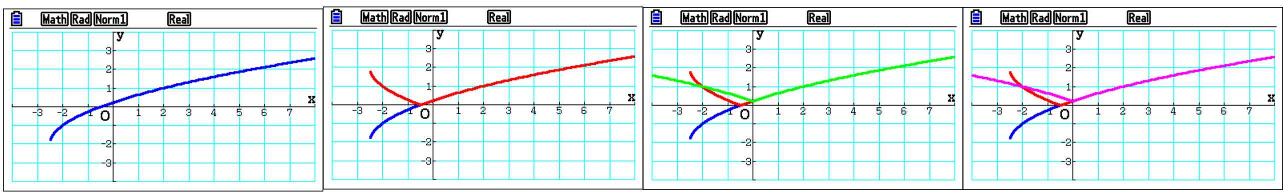
i



ii



iii



Exercise 2.3.1

1. Find the equation of the given relation under the translation indicated.

f $x^2+y = 2 ; \begin{pmatrix}0\\-2\end{pmatrix}$ g $xy = 8 ; \begin{pmatrix}4\\0\end{pmatrix}$ h $xy = 8 ; \begin{pmatrix}0\\-1\end{pmatrix}$

i $x^2+y^2 = 4 ; \begin{pmatrix}-1\\0\end{pmatrix}$ j $xy^2 = 9 ; \begin{pmatrix}3\\0\end{pmatrix}$ k $xy^2 = 9 ; \begin{pmatrix}0\\-3\end{pmatrix}$

l $x+y^2 = 4 ; \begin{pmatrix}4\\0\end{pmatrix}$

6 On the same set of axes sketch the graphs of:

e $y = x^3-8$ and $y = (x-8)^3$ f $y = (x+1)^3$ and $y = x^3+1$

g $y = \frac{1}{(x-2)}$ and $y = \frac{1}{x}-2$ h $y = \frac{1}{(x+3)}$ and $y = \frac{1}{x}+3$

i $y = \sqrt{x-2}$ and $y = \sqrt{x}-2$ j $y = \sqrt{x+4}$ and $y = \sqrt{x}+4$

7 Sketch the graphs of the following functions, making sure to include all axial intercepts and labelling the equations of asymptotes (where they exist).

i $y = (1+x)^2-1$ j $y = 1+\frac{1}{3+x}$ k $y = \frac{1}{(x-1)^2}-1$

l $y = -2+\frac{1}{(2-x)^2}$ m $y = 2-\frac{1}{3-x}$ n $y = 8-(2-x)^3$

o $y = -2+\sqrt{x-4}$

8 Find the vector translation necessary for the following mappings.

a $x^2 \mapsto x^2+4$ b $x^3 \mapsto x^3-2$ c $\sqrt{x} \mapsto \sqrt{x+1}$

d $\frac{1}{x} \mapsto \frac{1}{x-2}$ e $x^4 \mapsto (x+2)^4$ f $\frac{1}{x^2} \mapsto \frac{1}{x^2}-4$

g $x^3 \mapsto (x-2)^3-2$ h $\frac{1}{x} \mapsto 3+\frac{1}{x+2}$ i $x^2 \mapsto 2+(x-4)^2$

j $\sqrt{x} \mapsto 3+\sqrt{x-2}$ k $\frac{1}{x^3} \mapsto \frac{1}{(x-3)^3}-1$ l $f(x) \mapsto h+f(x+k)$

m $x^2-4 \mapsto (x+2)^2$ n $x^3+1 \mapsto (x-1)^3$ o $\frac{1}{x}-2 \mapsto \frac{1}{x+1}$

9 Express, in terms of $f(x)$, the transformation(s) required to map $f(x)$ to $g(x)$.

a $f(x) = x^2 , g(x) = x^2-2x+2$

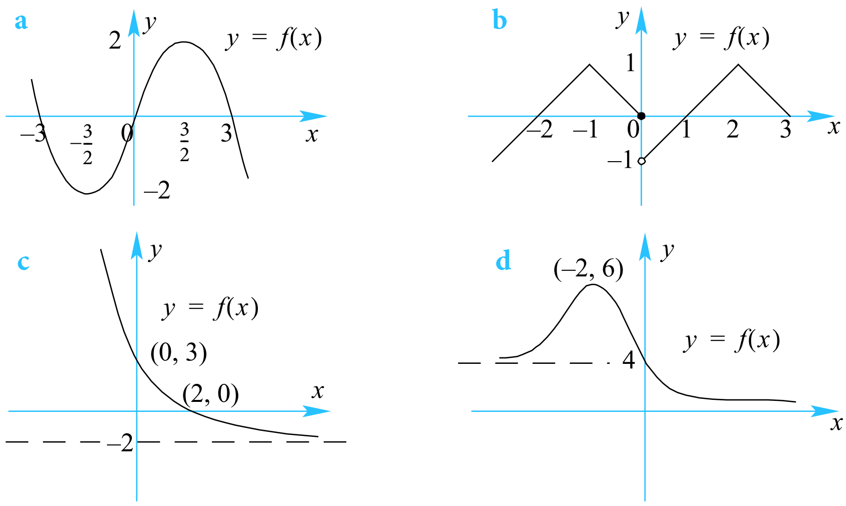
b $f(x) = x^2 , g(x) = x^2+4x$

c $f(x) = x^3 , g(x) = x^3-6x^2+12x-8$

d $\quad f(x) = \frac{1}{x^2},\ g(x) = \frac{1}{x^2 - 2x + 1} + 1$

e $\quad f(x) = x^3,\ g(x) = x^3 - 3x^2 + 3x + 2$

10 Consider the relations shown below.



Sketch the following.

i $\quad y = f(x+2) - 2$ $\qquad$ ii $\quad y = f(x-2) - 4$

iii $\quad y = 3 + f(x-3)$ $\qquad$ iv $\quad y = 1 + f(x+1)$

11. Express, in terms of $f(x)$, the equation of the graph represented in Figure 2, given that the graph in Figure 1 has the equation $y = f(x)$, $-1 \le x \le 1$.

Exercise 2.3.2

5. Describe the transformation(s) under the following mappings.

a $|x| \mapsto |2x| + 1$ b $x^2 \mapsto \frac{1}{2}(x-2)^2 - 3$ c $\frac{1}{x} \mapsto \frac{1}{2x-1}$

d $x^3 \mapsto (3x-2)^3$ e $x^4 \mapsto \frac{1}{2}(4x-2)^4 - 2$ f $\sqrt{x} \mapsto \frac{1}{2}\sqrt{8x} + 2$

6. Consider the function $g(x) = \begin{cases} x^2 & \text{if } x \geq 2 \\ 6 - x & \text{if } x < 2 \end{cases}$.

Find an expression for:

i $f(x) = g(x+2)$ ii $h(x) = g(x) - 3$ iii $h(x) = 2g(x)$

iv $k(x) = g(2x)$ v $k(x) = g(2x-1)$ vi $f(x) = \frac{1}{2}g(4x+2)$

On separate sets of axes, sketch the graphs of each of the functions in part a.

7. Given the relation , sketch the graphs of:

$$f(x) = \begin{cases} -\sqrt{4-(x-2)^2} & \text{if } 1 < x \leq 4 \\ \sqrt{3x} & \text{if } \quad x \leq 1 \\ -\sqrt{3x} & \text{if } \quad x \leq 1 \\ \sqrt{4-(x-2)^2} & \text{if } 1 < x \leq 4 \end{cases}$$

a $y = \frac{1}{2}f(x)$ b $y = f\left(\frac{1}{2}x\right)$

8. Given the function $f(x) = \sqrt{x}$, sketch the graphs of:

a $y = af(x), a > 0$ b $y = f(ax), a > 0$

c $y = bf(x+b)$, $b > 0$ d $y = \frac{1}{a}f(a^2x), a \neq 0$

9. Given the function $f(x) = \frac{1}{x^2}$, sketch the graphs of:

a $y = bf(\sqrt{ax}) - a$, a, b >0 b $y = bf(\sqrt{ax}) - \frac{a}{b}$, $a, b > 0$

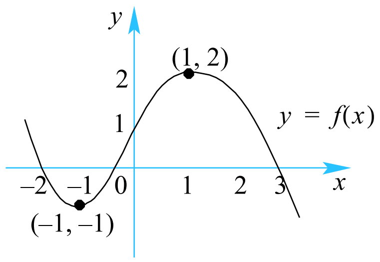
Exercise 2.3.3

1. Sketch the graphs of:

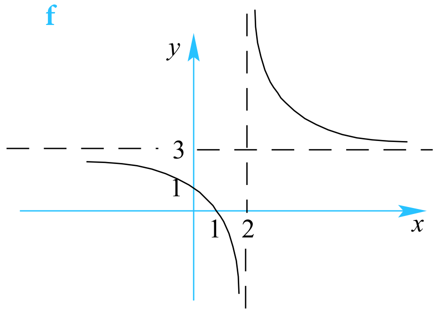
i $y = f(-x)$ ii $y = -f(x)$

for each of the following.

e



f



3. Sketch the graphs of the following.

n $f(x) = \frac{1}{2}\sqrt{2 - \frac{1}{4}x}$

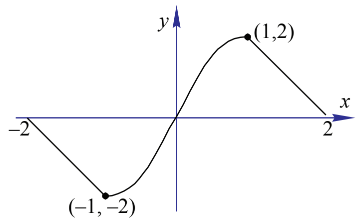
o $f(x) = |2 - x|$

p $f(x) = 2(1 - \sqrt{x-2})$

q $f(x) = 4 - |8 - x^3|$

r $f(x) = \dfrac{2}{2 - \sqrt{x}}$

4. The graph of $y = f(x)$ is shown opposite. Use it to sketch the graphs of:



i $y = 2 - f\left(\frac{1}{2}x\right)$ j $y = -2 + f(1 - x)$

5. Sketch the graphs of the following functions relative to the graph of:

a $f(x) = \frac{1}{x}$ i $y = f(ax) + b,\ a > 1,\ b < 0$ ii $y = bf(a - x),\ b > 0,\ a > 0$

b $f(x) = \frac{1}{x^2}$. i $y = bf(\sqrt{ax}) - a,\ a > 1,\ b < 0$ ii $y = f\left(\frac{x}{b}\right) - a,\ a > 0,\ b < 0$

c $f(x) = \sqrt{x}$ i $y = af(a^2 - ax),\ a > 0$ ii $y = af(a^2 - ax),\ a < 0$

6. Consider the following transformations.

A: Reflection about x-axis.

B: Reflection about y-axis

C: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

D: Squash by factor 2 along x-axis

E: Stretch by factor 3 along y-axis

Sketch the graph of $f(x)$ under the transformations in their given order

a $f(x) = |x-2|$, A; C.

b $f(x) = |x-2|$, C; A.

c $f(x) = |x^2-2x|$, D; B; C.

d $f(x) = |x^2-2x|$, C; D; B.

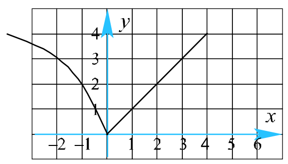
e $f(x) = x^3$, A; E; C.

f $f(x) = x^3$, E; C; A.

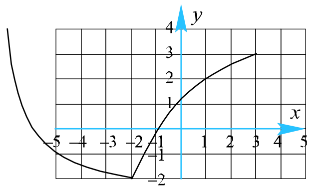
Exercise 2.3.1

1	a	$y = (x-4)^2$	b	$y = (x+2)^2$	c	$y = x^2+5$
	d	$(x-2)^2+y = 2$	e	$x^2+y = 4$	f	$x^2+y = 0$
	g	$y = \frac{8}{x-4}, x \neq 4$	h	$y = \frac{8}{x}-1, x \neq 0$	i	$(x+1)^2+y^2 = 4$
	j	$y^2 = \frac{9}{x-3}, x \neq 3$	k	$(y+3)^2 = \frac{9}{x}, x \neq 0$	l	$x+y^2 = 8$

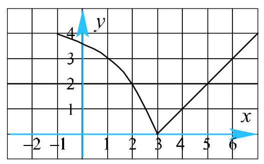
2 a i



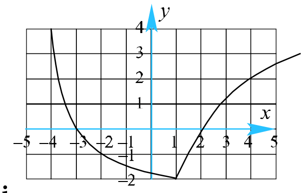
bi



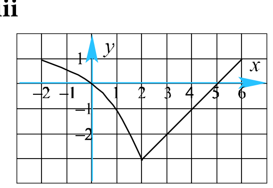
a ii



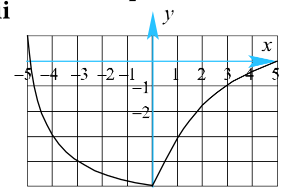
b ii



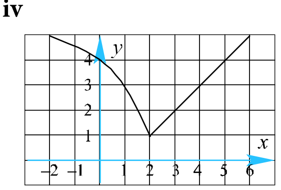
a iii



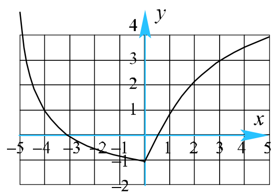
b iii



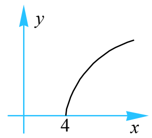
a iv



b iv



3 a



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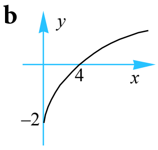
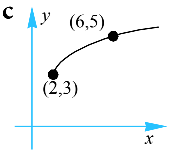
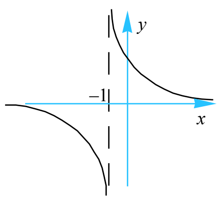


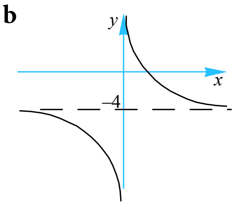
c



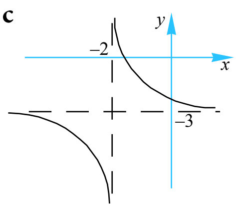
4 a



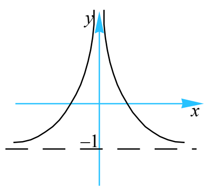
b



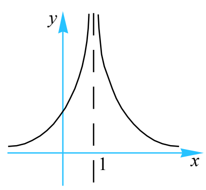
c



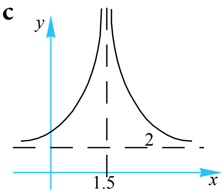
5 a



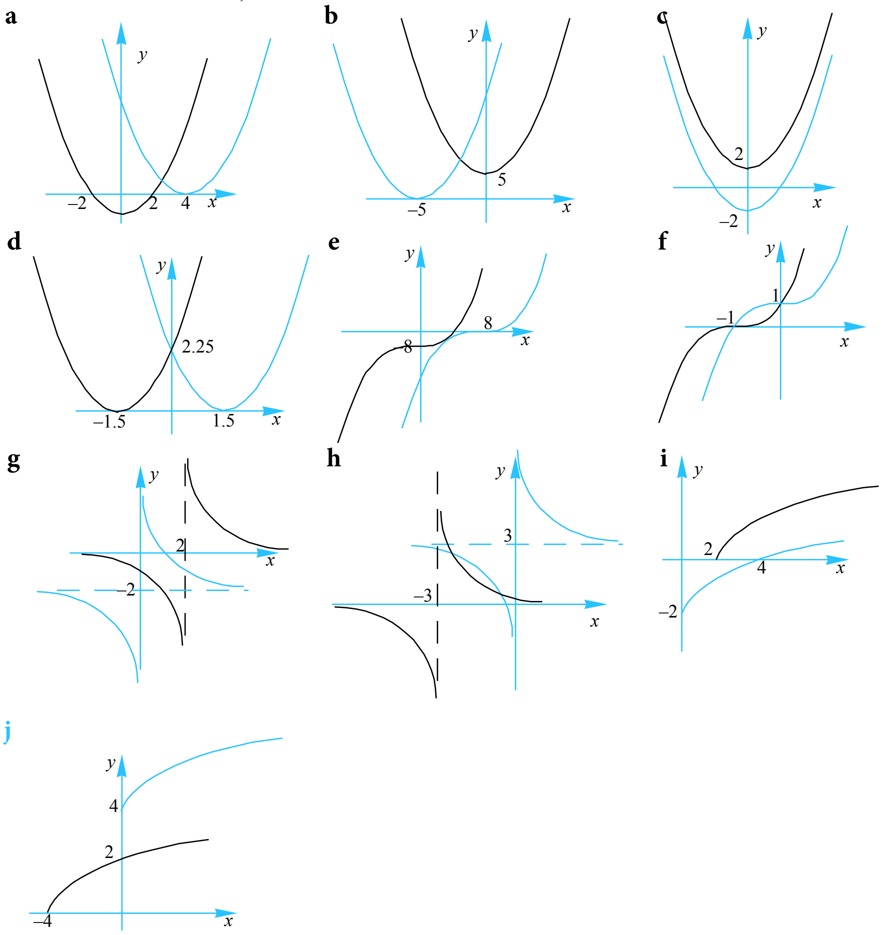
b



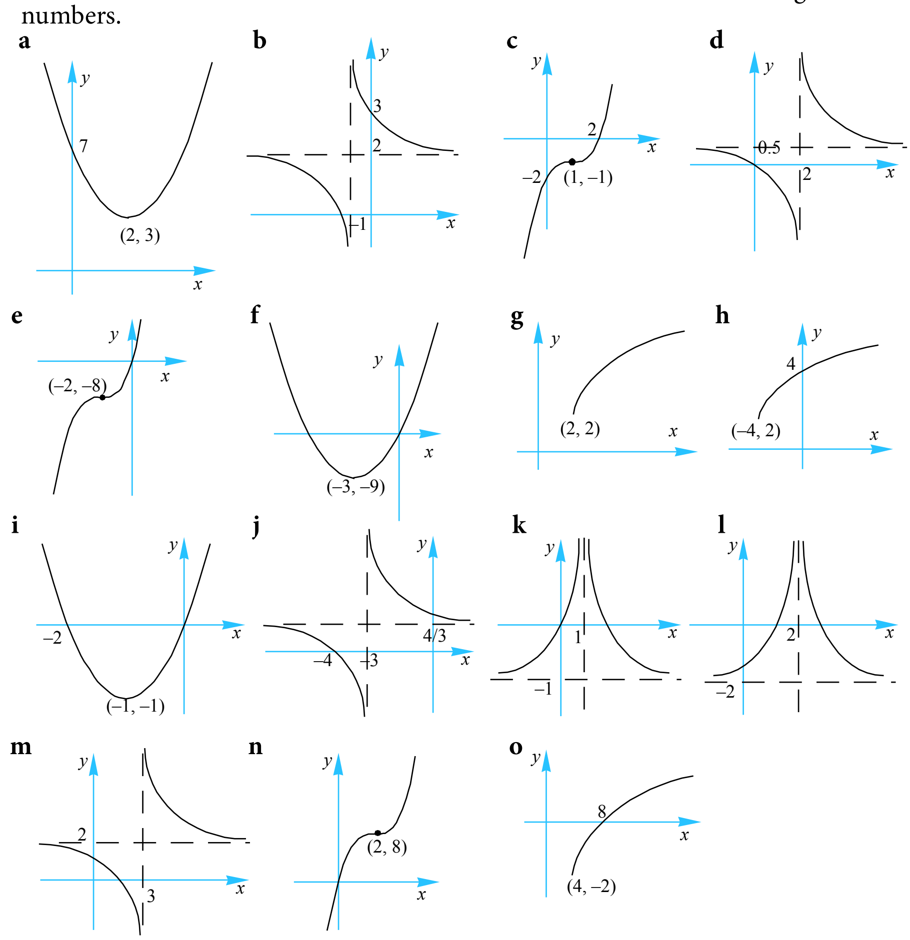
c



6 First function in black, second function in blue



7 Note: coordinates were asked for. We have labelled most of these with single numbers.



8 a $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ b $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ c $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ d $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

e $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ f $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ g $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ h $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

i $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ j $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ k $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ l $\begin{pmatrix} -k \\ h \end{pmatrix}$

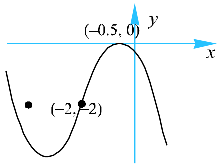
m $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ n $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ o $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

9 a $g(x) = f(x-1)+1$ b $g(x) = f(x+2)-4$

c $g(x) = f(x-2)$ d $g(x) = f(x-1)+1$

e $g(x) = f(x-1)+3$

10 a i



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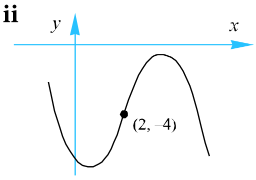

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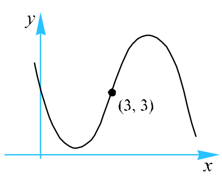

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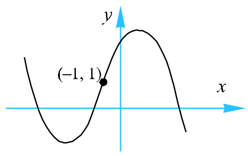
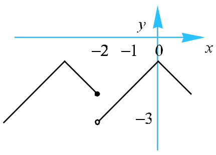


b i



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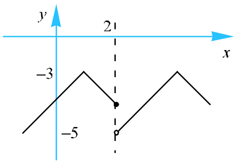

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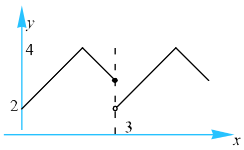

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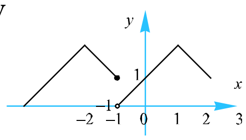
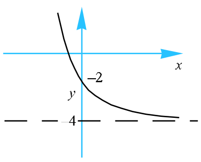


c i



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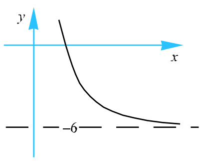

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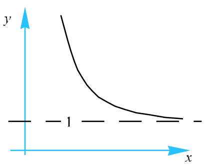

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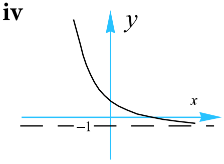
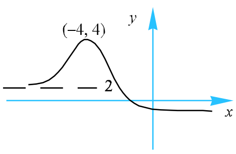


d i



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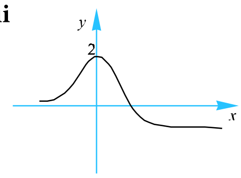

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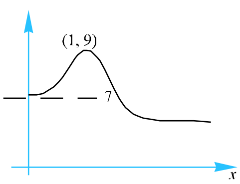

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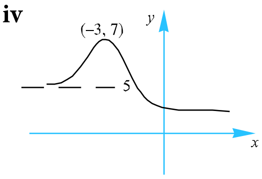
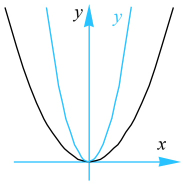


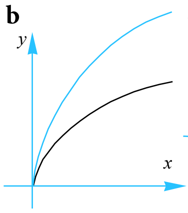
11 $y = \begin{cases} f(x+2)+2, & -3 \le x \le -1 \\ f(x+4)+2, & -5 \le x \le -3 \end{cases}$

Exercise 2.3.2

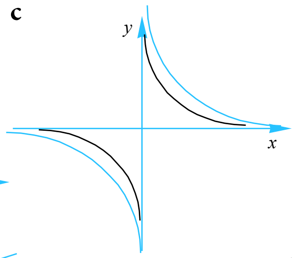
1 a



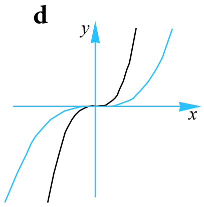
b



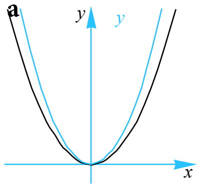
c



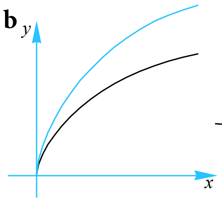
d



2 a



b



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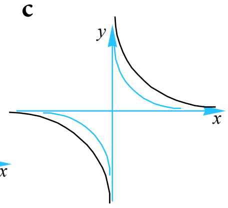
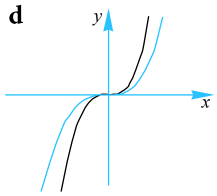
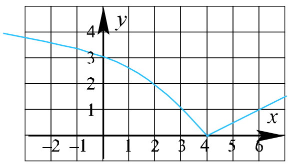


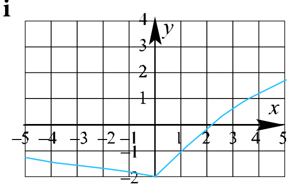
d



3a i



b i



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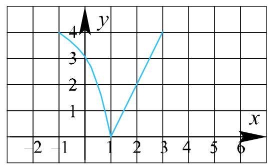

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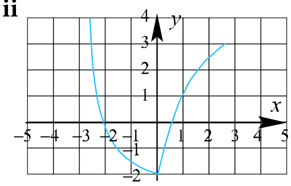

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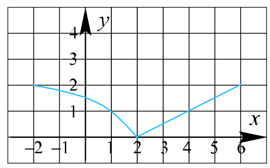

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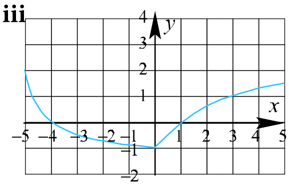

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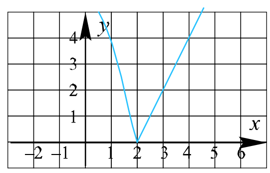

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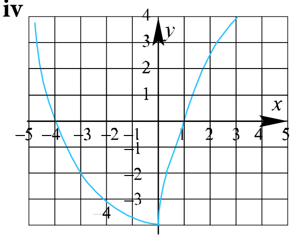
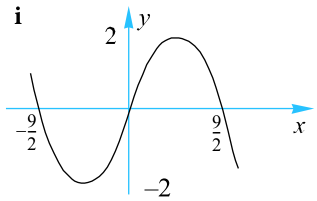


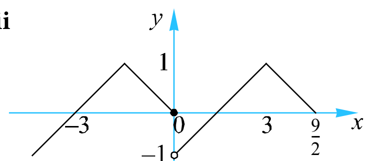
4 a

$y = f\left(\frac{2}{3}x\right)$

i



ii



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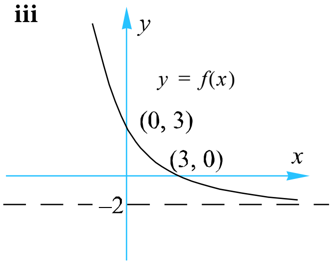

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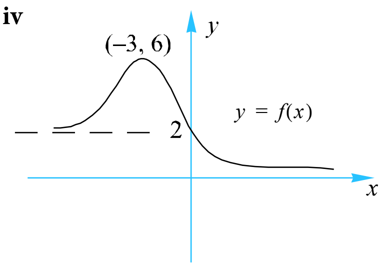
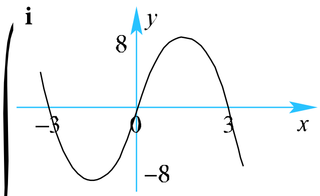


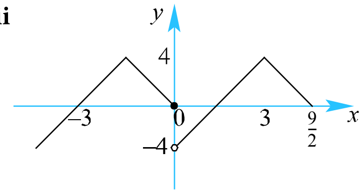
b

$y = 4f(x)$

i



ii



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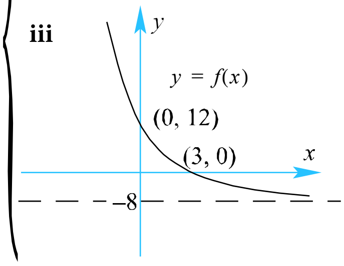

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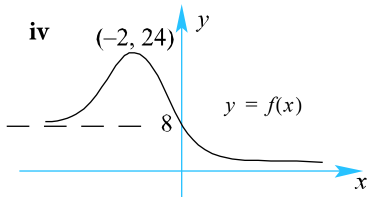


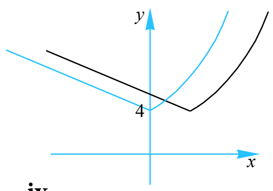
5 a $f(x) = |x| \quad y = f(2x) + 1$ b $f(x) = x^2 \quad y = \frac{1}{2}f(x-2) - 3$

c $f(x) = \frac{1}{x} \quad y = \frac{1}{2}f\left(x - \frac{1}{2}\right)$ d $f(x) = x^3 \quad y = 27f\left(x - \frac{2}{3}\right)$

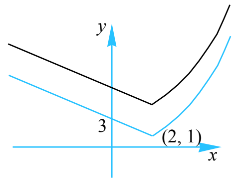
e $f(x) = x^4 \quad y = 128f\left(x - \frac{1}{2}\right) - 2$ f $f(x) = \sqrt{x} \quad y = \sqrt{2}f(x) + 2$

6

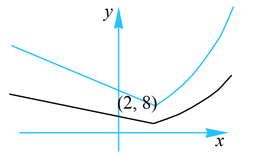
i $f(x) = \begin{cases} (x+2)^2 & \text{if } x \ge 0 \\ 4 - x & \text{if } x < 0 \end{cases}$



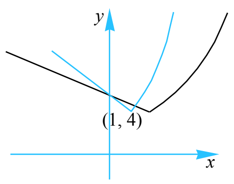
ii $h(x) = \begin{cases} x^2 - 3 & \text{if } x \ge 2 \\ 3 - x & \text{if } x < 2 \end{cases}$



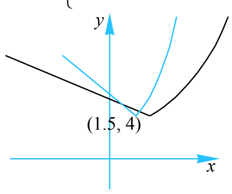
iii $h(x) = \begin{cases} 2x^2 & \text{if } x \ge 2 \\ 12 - 2x & \text{if } x < 2 \end{cases}$



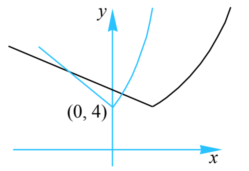
iv $k(x) = \begin{cases} 4x^2 & \text{if } x \ge 1 \\ 6 - 2x & \text{if } x < 1 \end{cases}$



v $k(x) = \begin{cases} (2x-1)^2 & \text{if } x \ge \frac{3}{2} \\ 7 - 2x & \text{if } x < \frac{3}{2} \end{cases}$



iv $f(x) = \begin{cases} \frac{1}{2}(4x+2)^2 & \text{if } x \ge 0 \\ 2 - 2x & \text{if } x < 0 \end{cases}$



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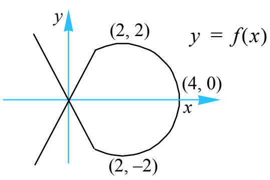

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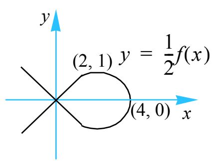

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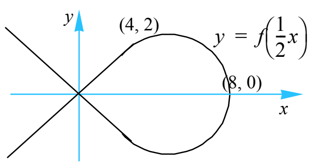

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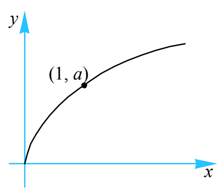

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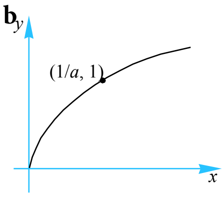

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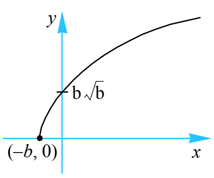

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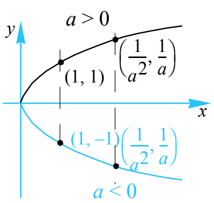

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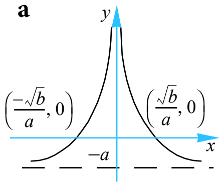

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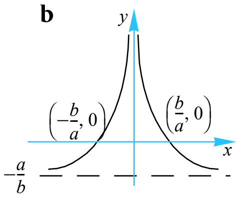

Exercise 2.3.3

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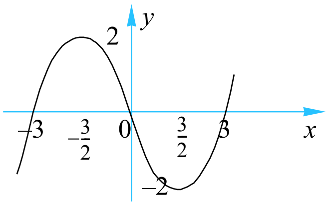

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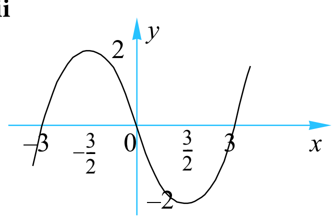

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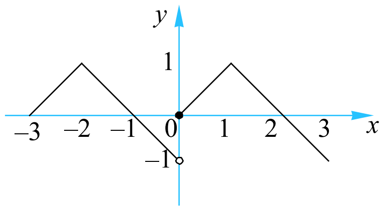

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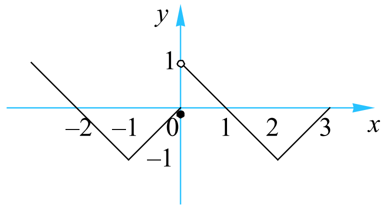

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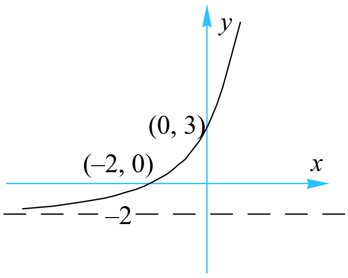

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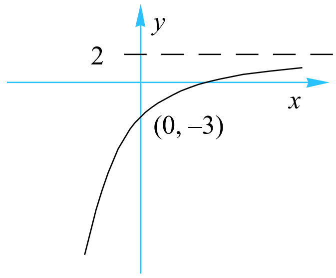

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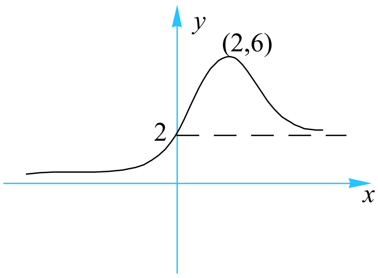

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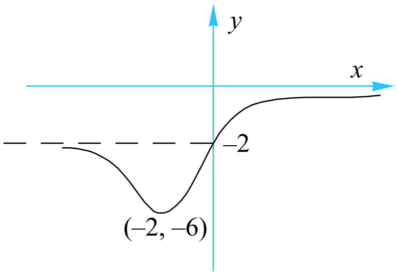

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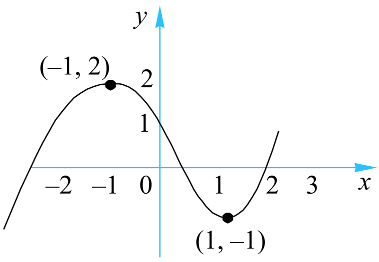

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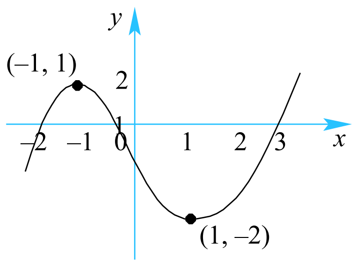

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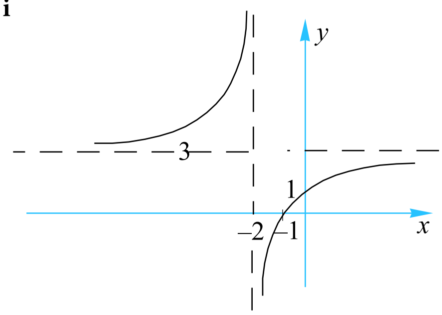

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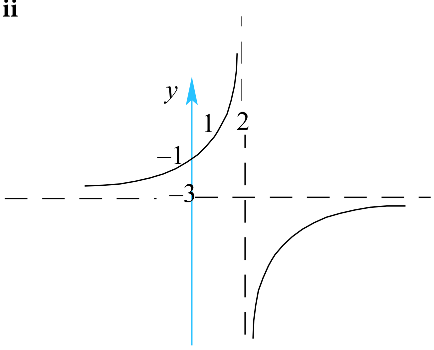
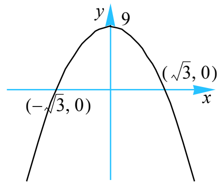


2 a $y = -f(x)$ b $y = f(-x)$ c $y = f(x+1)$

d $y = f(2x)$ e $y = 2f(x)$

3 a


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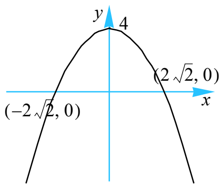

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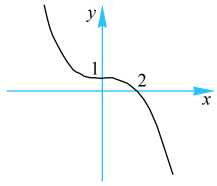

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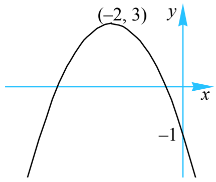

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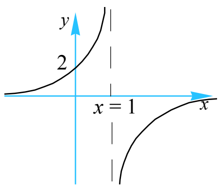

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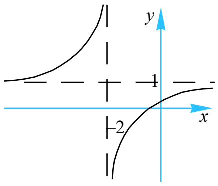

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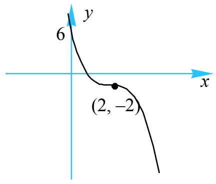

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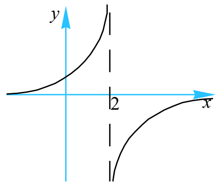

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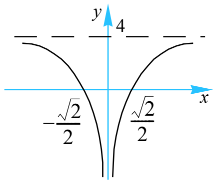

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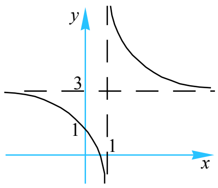

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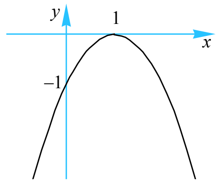

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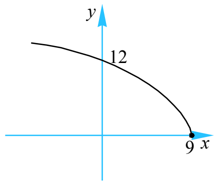

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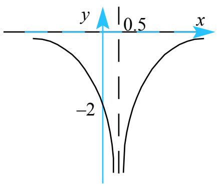

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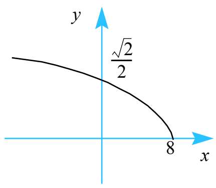

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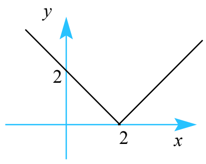

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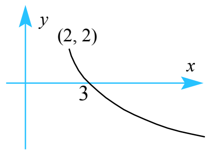

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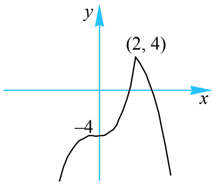

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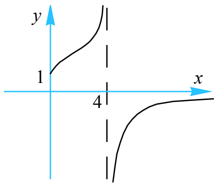

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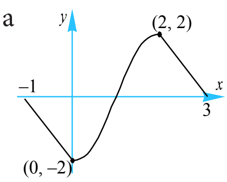

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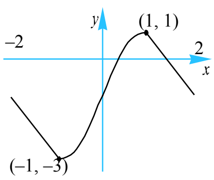

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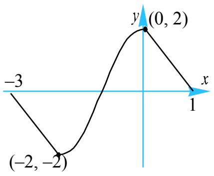

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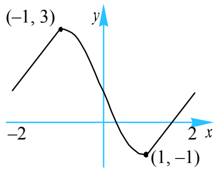

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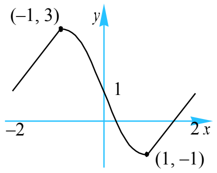

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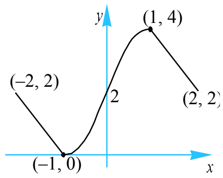

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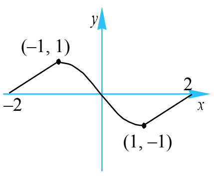

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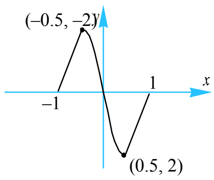

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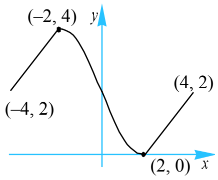

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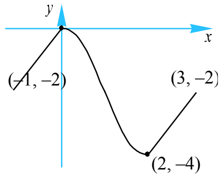

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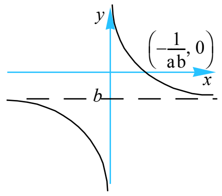

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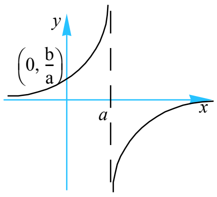

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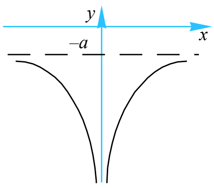

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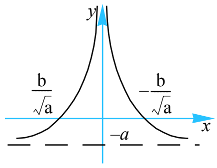

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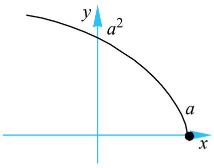

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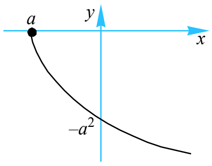

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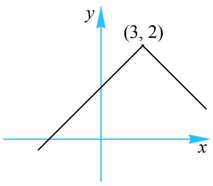

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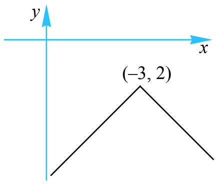

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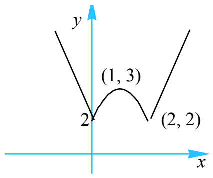

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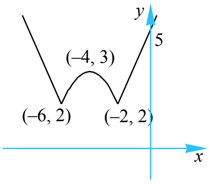

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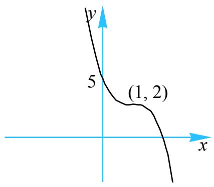

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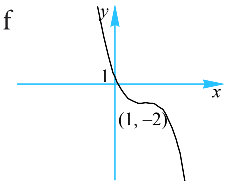


Exercise 2.4.1

6. Consider the function $f(x) = \dfrac{2-x}{2+x}$.

a i Find the coordinates of the intercepts with the axes.

ii Determine the equations of the asymptotes of f.

iii Hence, sketch the graph of f.

iv Determine the domain and range of f.

b Find f^{-1} , the inverse function of f.

b Deduce the graph of $(f(x))^2$.

7. Express $\dfrac{8x-5}{x-3}$ in the form $A + \dfrac{B}{x-3}$, where A and B are integers.

a Hence, state the equations of the vertical and horizontal asymptotes of the function $f(x) = \dfrac{8x-5}{x-3}$.

b Sketch the graph of $f(x) = \dfrac{8x-5}{x-3}$ and use it to determine its range.

8. On different sets of axes, sketch the graphs of $f(x) = 2 + \dfrac{1}{x}$ and $g(x) = \dfrac{1}{f(x)}$, stating their domains and ranges.

9. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = 2x + \frac{1}{x}, x \neq 0$

b $g(x) = \frac{1}{2}x + \frac{1}{x^2}, x \neq 0$

c $g(x) = -x + \frac{1}{x}, x \neq 0$

d $f(x) = x - \frac{1}{x}, x \neq 0$

10. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $h(x) = x^2 + \frac{2}{x}, x \neq 0$

b $f(x) = x^2 + \frac{1}{x^2}, x \neq 0$

c $g(x) = x - \frac{1}{x^2}, x \neq 0$

d $f(x) = x^3 + \frac{3}{x}, x \neq 0$

11. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = x + 3 + \frac{2}{x}, x \neq 0$

b $f(x) = -x + \frac{1}{x} + 2, x \neq 0$

c $g(x) = 2x + \frac{1}{x^2} - 2, x \neq 0$

d $f(x) = \frac{x^2 + 2x - 2}{x}, x \neq 0$

12. a For the function $f(x) = 3 + \frac{1}{1-x} - x$:

i determine all axial intercepts and the coordinates of its stationary points.

ii write down the equation of all the asymptotes.

b Sketch the graph of $y = f(x)$ clearly labelling all the information from part a.

13. Sketch the graphs of: a $f(x) = \dfrac{x^2 - x - 1}{x - 2}, x \neq 2$. b $g(x) = \dfrac{(x+2)^2(x-1)}{x^2}, x \neq 0$.

14. Sketch the graphs of the following functions.

a $f(x) = \dfrac{2x-3}{x^2-3x+2}$ b $y = \dfrac{x^2+2x}{x^2+4}$ c $y = \dfrac{x^4+1}{x^2+1}$.

15. Sketch the graph of $f(x) = \dfrac{x+1}{\sqrt{x-1}}$, clearly identifying all asymptotes and turning points.

Exercise 2.4.2

7. On the same set of axes, sketch the graphs of $f(x) = 5\times 5^{-x}$ and $g(x) = 5^x - 4$.

   Find: i $\{(x,y) : f(x) = g(x)\}$ ii $\{x : f(x) > g(x)\}$.

8. Find the range of the following functions.

   a f: $]0,\infty[\mapsto \mathbb{R}$, where $f(x) = e^{-(x+1)} + 2$.

   b $g(x) = -2\times e^x + 1, x \in]-\infty,0]$.

   c $x \mapsto xe^{-x} + 1, x \in [-1,1]$

9. a Sketch the graph of $f(x) = |2^x - 1|$, clearly labelling all intercepts with the axes and the equation of the asymptote.

   b Solve for x, where $|2^x - 1| = 3$.

10. Sketch the graphs of the following functions:

    a $f(x) = |1 - 2^x|$ b $g(x) = |4^x - 2|$ c $h(x) = 1 - |2^x|$

11. Sketch the graphs of the following functions.

    a $f(x) = 1 - 2^{-|x|}$ b $g(x) = -4 + 2^{|x|}$ c $h(x) = |3^{-|x|} - 3|$

12. Sketch the graphs of the following functions and find their range.

    a $f(x) = \begin{cases} 2^x, & x < 1 \\ 3, & x \geq 1 \end{cases}$ b $f(x) = \begin{cases} 3 - e^x, & x > 0 \\ x + 3, & x \leq 0 \end{cases}$

    c $f(x) = \begin{cases} \frac{2}{x+1}, & x \geq 1 \\ 3 - 2^{2-x}, & x < 1 \end{cases}$ d $g(x) = \begin{cases} 4 - 3^{-|x|}, & -1 < x < 1 \\ 4 - \frac{1}{3}|x|, & 1 \leq |x| \leq 12 \end{cases}$

13. Sketch the graphs of the following, and hence state the range in each case.

    a f: $\mathbb{R}\mapsto\mathbb{R}\ y = 2^x + \left(\frac{1}{2}\right)^x$ b f: $\mathbb{R}\mapsto\mathbb{R}\ y = 3^x + \left(\frac{1}{3}\right)^x$

    c f: $\mathbb{R}\mapsto\mathbb{R}\ y = 2^x - \left(\frac{1}{2}\right)^x$ d f: $\mathbb{R}\mapsto\mathbb{R}\ y = \left|2^x - \left(\frac{1}{2}\right)^x\right|$

14. Sketch the graph of the functions.

    a $g(x) = 2^{(x-a)}, a > 0$ b $h(x) = 2^x - a, 0 < a < 1$

    c $f(x) = 2\times a^x - 2a, a > 1$ d $f(x) = 2\times a^x - 2a, 0 < a < 1$

    e $g(x) = a - a^x, a > 1$ f $h(x) = -a + a^{-x}, a > 1$

15. a On the same set of axes, sketch $f(x) = 2\times a^x$ and $g(x) = 4\times a^{-x}$ where a > 1.

    Hence, sketch the graph of the function $h(x) = a^x + 2a^{-x}$, where a > 1.

    b On the same set of axes, sketch $f(x) = x - a$ and $g(x) = a^{x+1}$, where $a > 1$.

    Hence, deduce the graph of $h(x) = (x-a)\times a^{x+1}$, where $a > 1$.

16. Sketch the graph of the following functions and determine their range.

a $f(x) = a^{-x^2}, a > 1$

b $f(x) = a^{-x^2}, 0 < a < 1$

c $g(x) = (a-1)^{-x}, a > 1$

d $h(x) = 2 - a^{-|x|}, a > 1$

e $f(x) = \dfrac{2}{a^x - 1}, a > 1$

f $g(x) = |a^{x^2} - a|, a > 1$

Exercise 2.4.3

6. Given the function $y = f(x)$, sketch the graphs of:

a $y = |f(x)|$ b $y = f(|x|)$

c $f(x) = \log_{10}(-x)$ d $f(x) = \ln\left(\frac{1}{x} - e\right)$ p

e $f(x) = 2 - \ln(ex - 1)$ f $f(x) = \log_2(x^2 - 2x)$

7. a On the same set of axes, sketch the graphs of $f(x) = \ln x - 1$ and $g(x) = \ln(x - e)$.

b Find $\{x : \ln x > \ln(x - e) + 1\}$.

8. Sketch the graphs of the following functions and find their ranges.

a $f(x) = \begin{cases} \log_{10}x, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$ b $f(x) = \begin{cases} \log_2(x^2 - 1), & |x| \geq 1 \\ 1 - x^2, & |x| < 1 \end{cases}$

c $f(x) = \begin{cases} 2 - \ln x, & x \geq e \\ \frac{x^3}{e^3}, & x < e \end{cases}$ d $g(x) = \begin{cases} 1 + \sqrt{x - 1}, & x > 1 \\ |\log_2 x| + 1, & 0 < x \leq 1 \\ 1 & x \leq 0 \end{cases}$

9. Sketch the graphs of the following functions.

a $f(x) = \log_{\frac{1}{2}}x$ b $f(x) = \log_{\frac{1}{2}}(x - 2)$ c $f(x) = \log_{\frac{1}{3}}x + 1$

10. Sketch the graph of the following functions, clearly stating domains and labelling asymptotes.

a $f(x) = 2\log_a(x - a), a > 1$ b $f(x) = -\ln(ax - e), a > e$

c $g(x) = |\log_{10}(10 - ax)|, 1 < a < 10$ d $g(x) = \ln|x - ae|, a > 1$

e $g(x) = |\ln|x - ae||, a > 1$ f $h(x) = \log_a\left(1 - \frac{x}{a}\right), 0 < a < 1$

11. Sketch the graph of $f(x) = \frac{1}{a}\log_a(ax - 1), 0 < a < 1$ clearly labelling its asymptote, and intercept(s) with the axes.

Hence, find $\left\{x : f(x) > \frac{1}{a}\right\}$.

12. Sketch the graph of: a $f(x) = \frac{\ln x}{x}, x > 0$ b $g(x) = \frac{x}{\ln x}, x > 0$

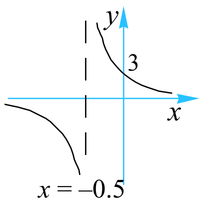
Given that $f(x) \leq e^{-1}$ for all real $x > 0$, state the range of $g(x)$.

Exercise 2.4.1

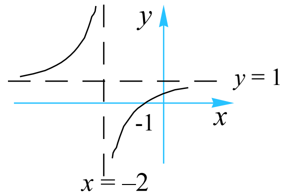
1 **a** $y = 2, x = -1$ **b** $y = 1, x = -\frac{1}{3}$ **c** $y = \frac{1}{2}, x = -\frac{1}{4}$

d $y = -1, x = -3$ **e** $y = 3, x = 0$ **f** $y = 5, x = 2$

3 a



b



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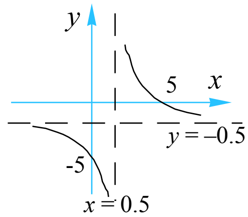

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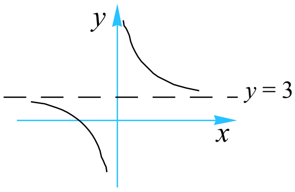

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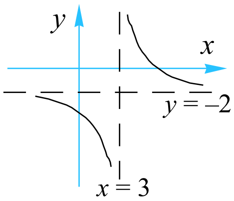

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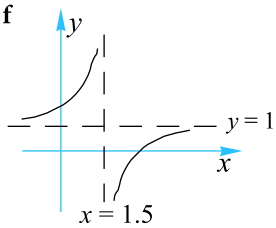
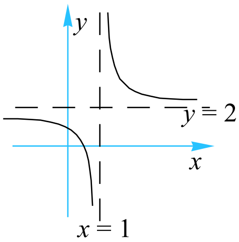


4 $a = 2, c = 4$

5 a



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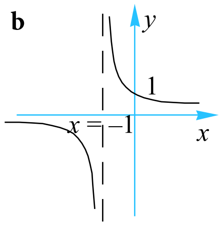
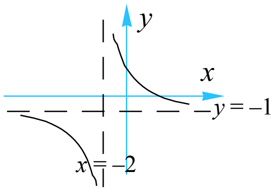
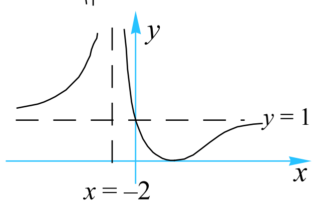


6 **a i** $(0, 1), (2, 0)$ **ii** $y = -1, x = -2$ **iii**

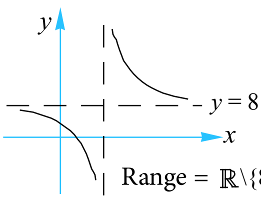


iv $d = \mathbb{R} \setminus \{-2\}$

b $f^{-1}: \mathbb{R}\setminus\{-1\} \mapsto \mathbb{R}$, where $f^{-1}(x) = \frac{2(1-x)}{(1+x)}$ **c**

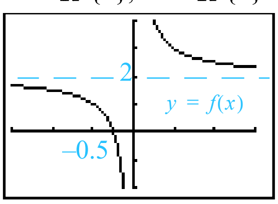


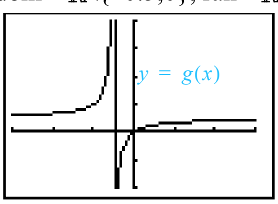
7 **a** $y = 8, x = 3$ **b**



Range = $\mathbb{R}\setminus\{8\}$

8 dom = $\mathbb{R}\setminus\{0\}$, ran = $\mathbb{R}\setminus\{2\}$ dom = $\mathbb{R}\setminus\{-0.5, 0\}$, ran = $\mathbb{R}\setminus\{0.5\}$





9 Asymptotes: **a** $y = 2x, x = 0$ **b** $y = \frac{1}{2}x, x = 0$ **c** $y = -x, x = 0$ **d** $y = x, x = 0$

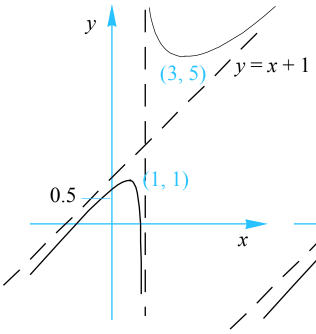
10 Asymptotes: **a** $y = x^2, x = 0$ **b** $y = x^2, x = 0$ **c** $y = x, x = 0$ **d** $y = x^3, x = 0$

11 Asymptotes: **a** $y = x+3, x = 0$ **b** $y = -x+2, x = 0$ **c** $y = 2x-2, x = 0$

d $y = x+2, x = 0$

12 **a i** (0, 4), (2, 0) **ii** $y = 3-x, x = 1$

13 **a**



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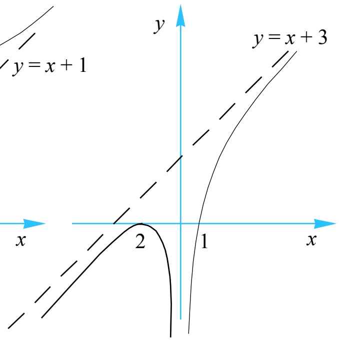
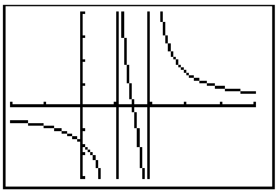


14 a



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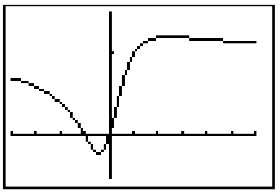

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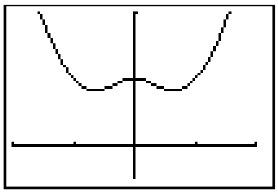

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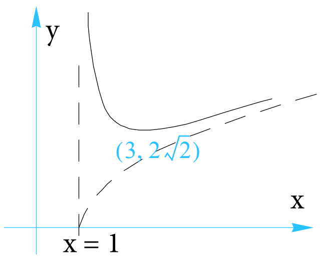
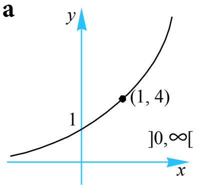


Exercise 2.4.2

1 a



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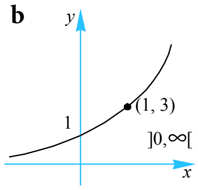
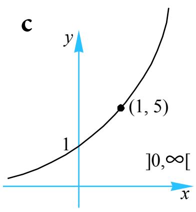
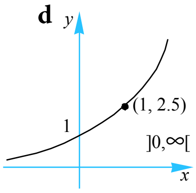


c



d



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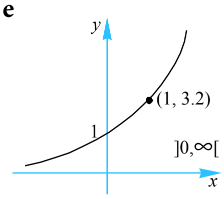

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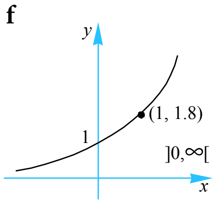

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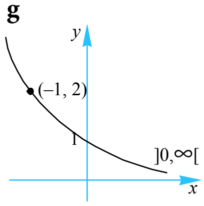

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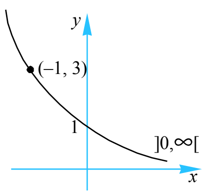

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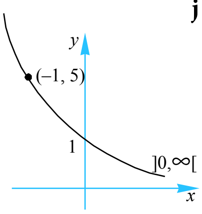

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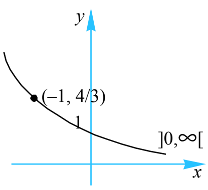

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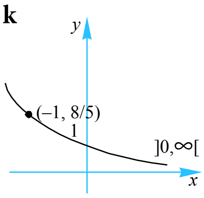
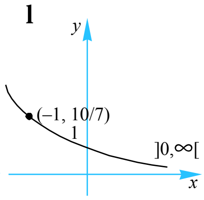
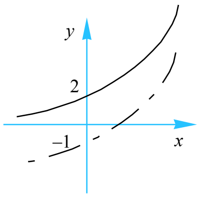


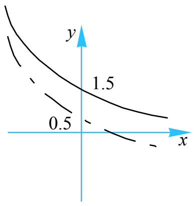
l



2 a

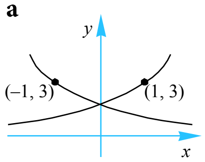


b

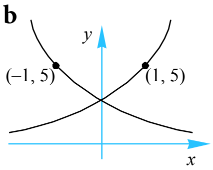


3 '*b*' has a dilation effect on $f(x) = a^x$ (along the *y* axis).

4 a



b



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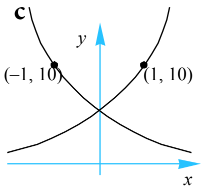
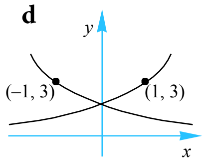
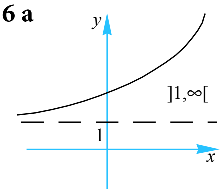


d



5 **a** [1,16] **b** [3,27] **c** [0.25,16] **d** [0.5,4] **e** [0.125,0.25] **f** [0.1,10]

6 a



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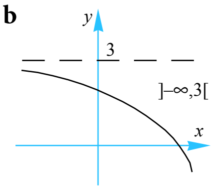

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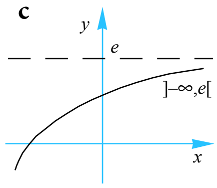

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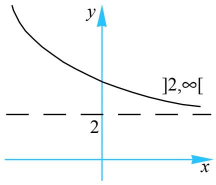
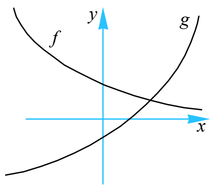


7 a –1. 5

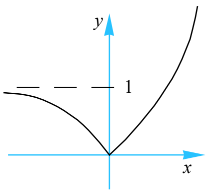
b



c i f = g: x = 1 **ii** f > g: x < 1

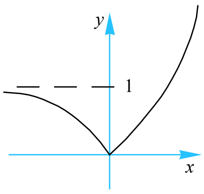
8 **a** $]2, 2 + e^{-1}[$ **b** $[-1, 1[$ **c** $[1 - e, 1 + e^{-1}]$

9 **a**



b 2

10 a



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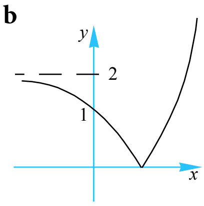

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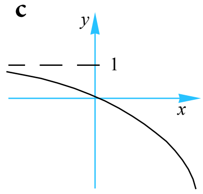
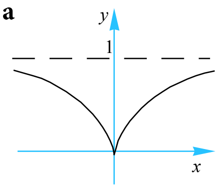
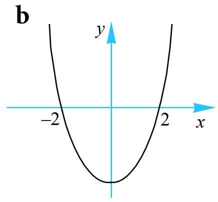


11 a



b



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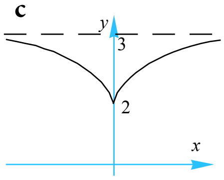
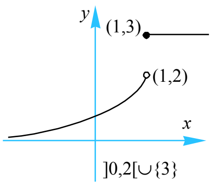


12 a



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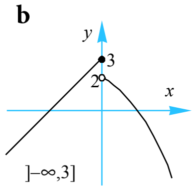

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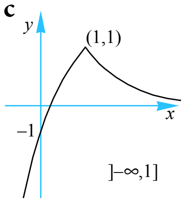

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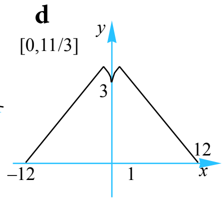
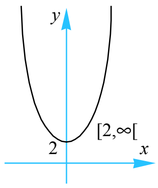


13 a



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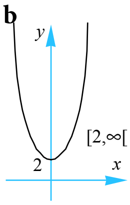

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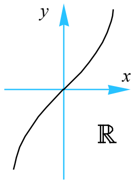

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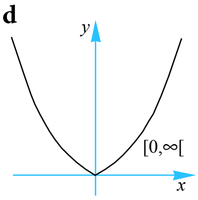

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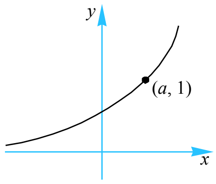

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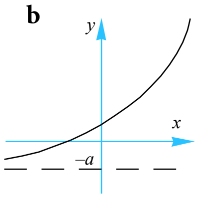

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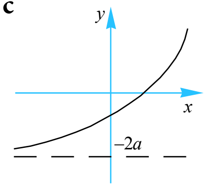

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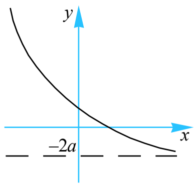

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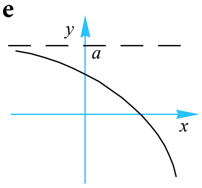

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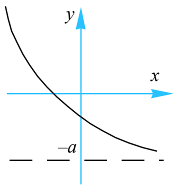

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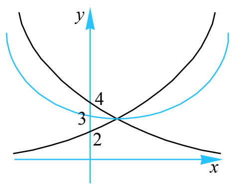

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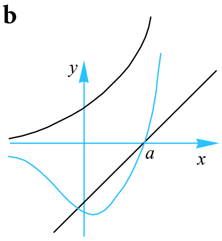

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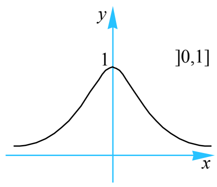

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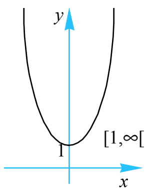

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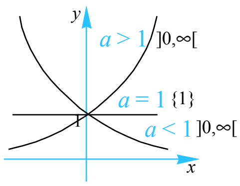

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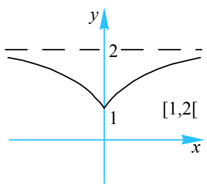

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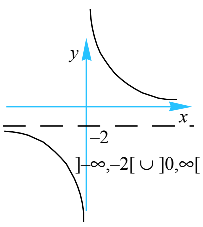

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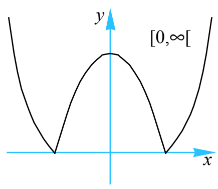

Exercise 2.4.3

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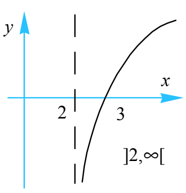

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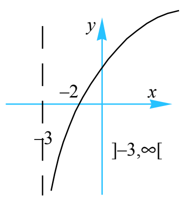

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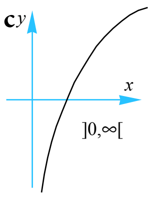

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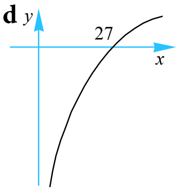

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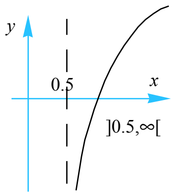

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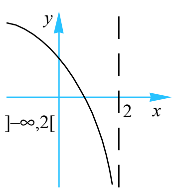

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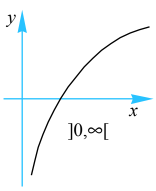

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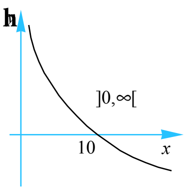

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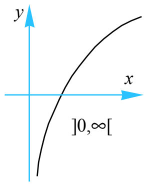

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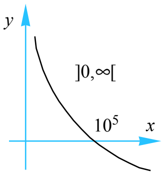

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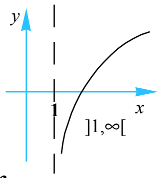

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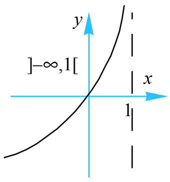

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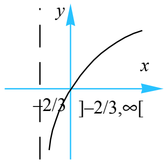

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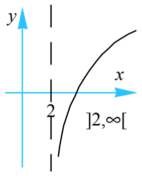

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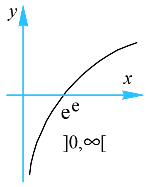

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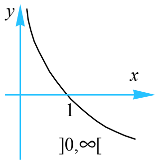

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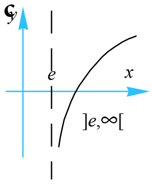

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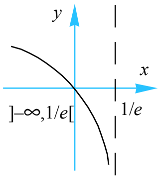

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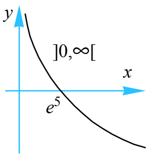

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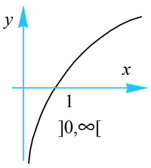

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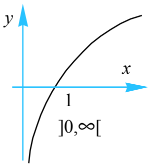

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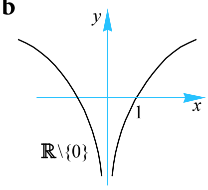

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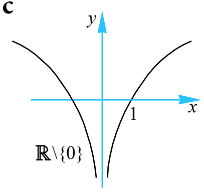

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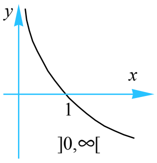

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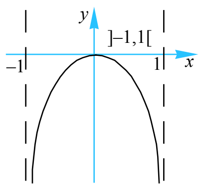

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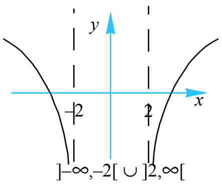

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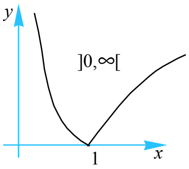

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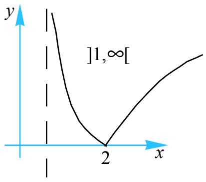

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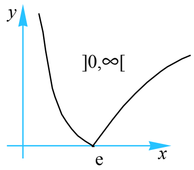

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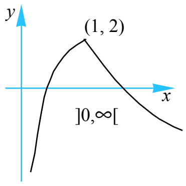

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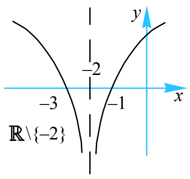

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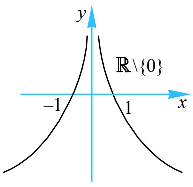

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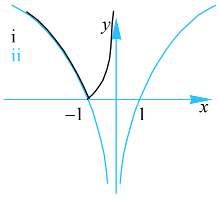

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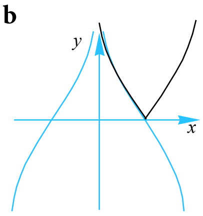

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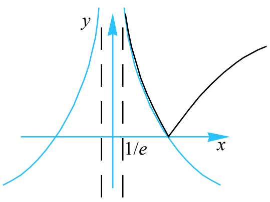

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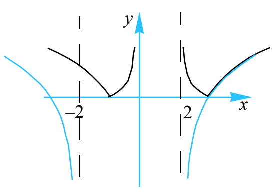

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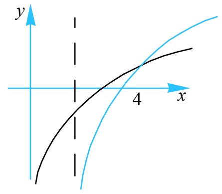
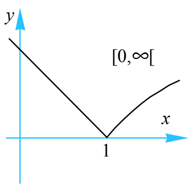


b $0 < x < \sim 4.3$

8 a



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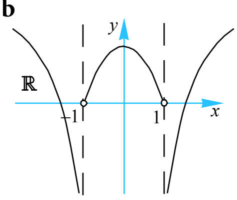

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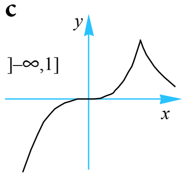

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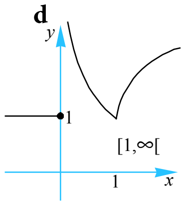

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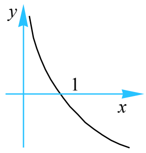

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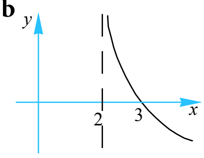

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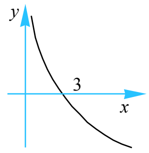

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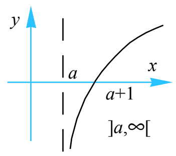

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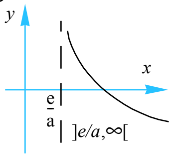

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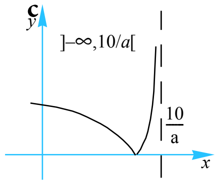

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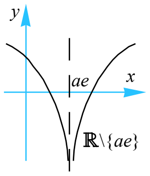

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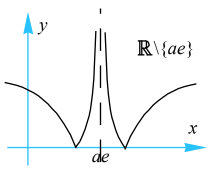

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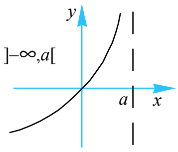

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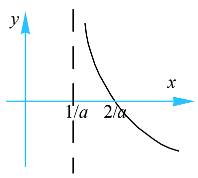
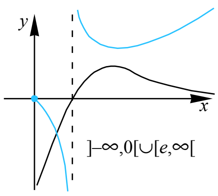


$$\left\{x : \frac{1}{a} < x < 1 + \frac{1}{a}\right\}$$

12



Exercise 2.5.1

1. Sketch the graphs of the following polynomials:

g $P(x) = -x^2(x-4)$

h $P(x) = (1-4x^2)(2x-1)$

i $T(x) = (x-1)(x-3)^2$

j $T(x) = \left(1-\frac{x}{2}\right)^2(x+2)$

k $P(x) = x^2(x+1)(2x-3)$

l $P(x) = 4x^2(x-2)^2$

m $P(x) = \frac{1}{2}(x-3)(x+1)(x-2)^2$

n $T(x) = -(x-2)(x+2)^3$

o $P(x) = (x^2-9)(3-x)^2$

p $T(x) = -2x(x-1)(x+3)(x+1)$

p $P(x) = x^4+2x^3-3x^2$

r $T(x) = \frac{1}{4}(4-x)(x+2)^3$

s $T(x) = -x^3(x^2-4)$

t $T(x) = (2x-1)\left(\frac{x}{2}-1\right)(x-1)(1-x)$

2. Sketch the graph of the following polynomials:

g $P(x) = -2x^4+3x^3+3x^2-2x$

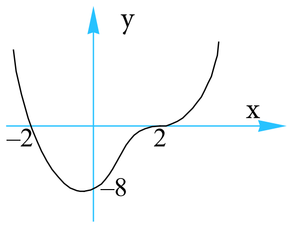
h $P(x) = 2x^4-3x^3-9x^2-x+3$

i $T(x) = x^4-5x^3+6x^2+4x-8$

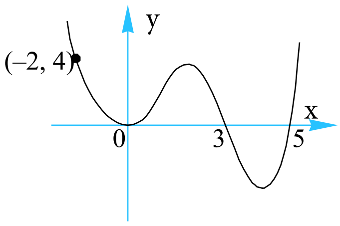
j $T(x) = x^4+2x^3-3x^2-4x+4$

5. Determine the equation of the following functions:

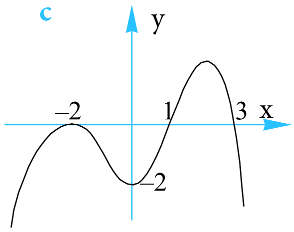
a



b



c



6. Sketch a graph of $f(x) = (x-b)(ax^2+bx+c)$ if $b > 0$ and

a $b^2-4ac = 0, a > 0, c > 0$

b $b^2-4ac > 0, a > 0, c > 0$

c $b^2-4ac < 0, a > 0, c > 0$

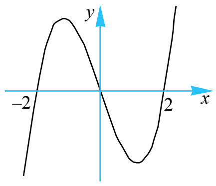
7.

a On the same set of axes sketch the graphs of $f(x) = (x-a)^3$ and $g(x) = (x-a)^2$. Find $\{(x, y) : f(x) = g(x)\}$.

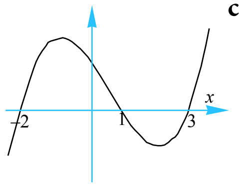
b Hence find $\{x: (x-a)^3 > (x-a)^2\}$.

Exercise 2.5.1

1 **a**



b



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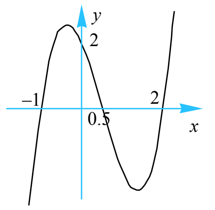
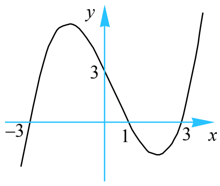


d



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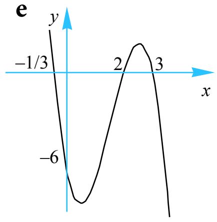

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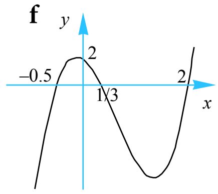

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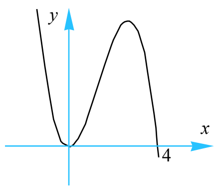

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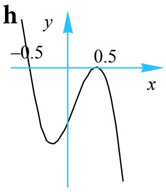

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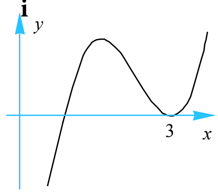

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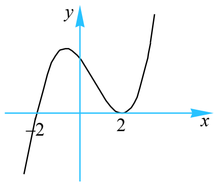

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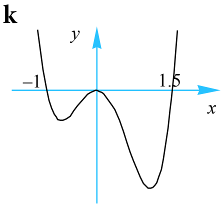

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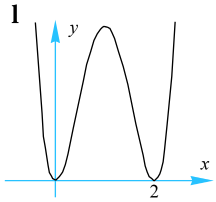

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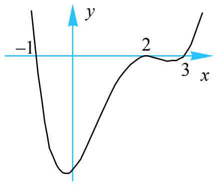

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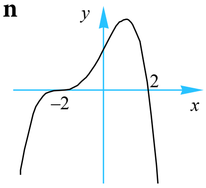
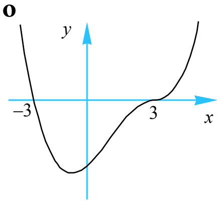
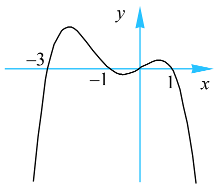


o



p



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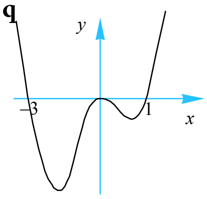

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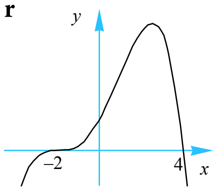

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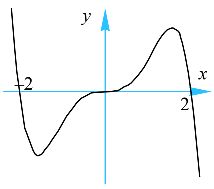

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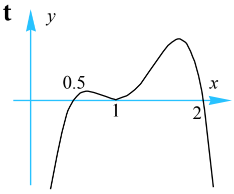
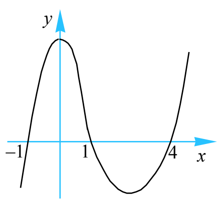
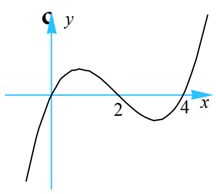


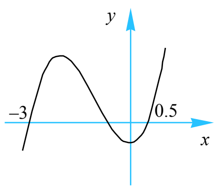
2 **a**



b

c





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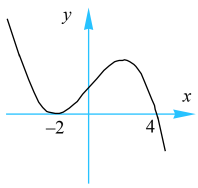

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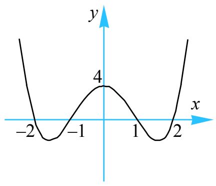

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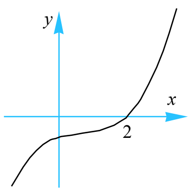

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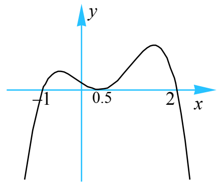

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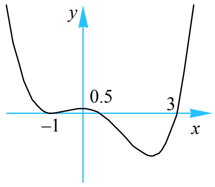

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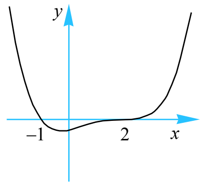

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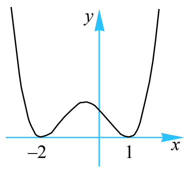
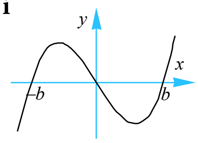


3 a i



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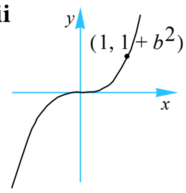
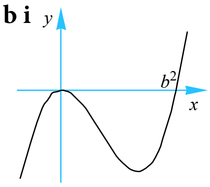


b i



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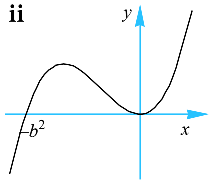


4

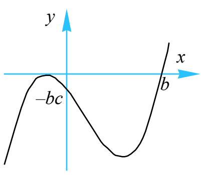
a $y = -\frac{1}{15}(x+3)(x-1)(x-5)$ **b** $y = \frac{1}{8}(x-2)^2(x+4)$ **c** $y = -\frac{3}{2}x^2(x-3)$

d $y = \frac{1}{30}(x+2)^2(75-29x)$ **e** $y = \frac{1}{6}(x+2)(4-3x)(x-3)$

f $y = -x^3 - x^2 + 2x + 8$

5 **a** $y = \frac{1}{2}(x+2)(x-2)^3$ **b** $y = \frac{1}{35}x^2(x-3)(x-5)$ **c** $y = -\frac{1}{6}(x+2)^2(x-1)(x-3)$

6 a



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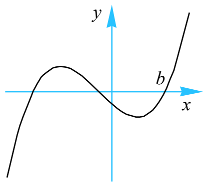

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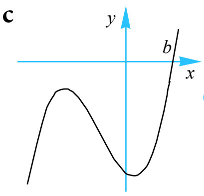
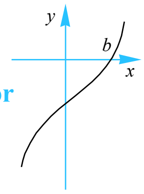
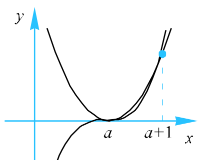


or



7 a



$\{(a, 0), (a+1, 1)\}$

b $\{x : x > a+1\}$

Exercise 2.6.1

1. By using a factorisation process, solve for the given variable.

e	$(3-u)(u+6) = 0$	f	$3x^2 + x - 10 = 0$
g	$3v^2 - 12v + 12 = 0$	h	$y(y-3) = 18$
i	$(x+3)(x+2) = 12$	j	$(2a-1)(a-1) = 1$

4. Use the quadratic formula to solve these equations.

i	$x^2 - 3x - 7 = 0$	**j**	$x^2 - 3x + 9 = 0$
k	$x^2 + 9 = 8x$	**l**	$4x^2 - 8x + 9 = 0$
m	$4x^2 = 8x + 9$	**n**	$5x^2 - 6x - 7 = 0$
o	$5x^2 - 12x + 1 = 0$	**p**	$7x^2 - 12x + 1 = 0$

Exercise 2.6.2

2. Given one of the roots of each of these equations, find the other.

g $4x^2 = 8x - 3$, $\alpha = \frac{1}{2}$ h $x^2 + 12 = 7x$, $\alpha = 3$ i $x^2 = 7x$, $\alpha = 0$

3. Generalize the results of this section to higher level polynomial equations.

4. *EXTENSION*: Not all quadratic equations have real roots. Even these equations have a sum of roots that is a real number. Explain this result.

Exercise 2.6.1

1	**a** -5	**b**	$4, 6$	**c** $-3, 0$		**d**	$1, 3$
	e $-6, 3$	**f**	$-2, \frac{5}{3}$	**g** 2		**h** $-3, 6$	
	i $-6, 1$	**j**	$0, \frac{3}{2}$				
2	**a** -1	**b**	$-7, 5$	**c** $-\frac{2}{5}, 3$		**d**	$-2, 1$
	e $-3, 1$	**f**	$4, 5$				
3	**a** $-1 \pm \sqrt{6}$	**b**	$3 \pm \sqrt{5}$	**c**	$1 \pm \sqrt{5}$		
	d $\frac{-1 \pm \sqrt{33}}{8}$	**e**	$\frac{9 \pm \sqrt{73}}{4}$	**f**	$\frac{1 \pm \sqrt{85}}{6}$		
4	**a** $\frac{3 \pm \sqrt{37}}{2}$	**b**	$\frac{5 \pm \sqrt{33}}{2}$	**c**	$\frac{3 \pm \sqrt{33}}{2}$		
	d $\frac{7 \pm \sqrt{57}}{2}$	**e**	$\frac{-7 \pm \sqrt{65}}{2}$	**f**	$-4, 2$		
	g $-1 \pm 2\sqrt{2}$	**h**	$\frac{-5 \pm \sqrt{53}}{2}$	**i**	$\frac{3 \pm \sqrt{37}}{2}$		
	j no real solutions			**k**	$4 \pm \sqrt{7}$		
	l no real solutions			**m**	$\frac{2 \pm \sqrt{13}}{2}$		
	n $\frac{3 \pm 2\sqrt{11}}{5}$	**o**	$\frac{6 \pm \sqrt{31}}{5}$	**p**	$\frac{6 \pm \sqrt{29}}{7}$		
5	**a** $-2 < p < 2$	**b**	$p = \pm 2$	**c**	$p < -2$ or $p > 2$		
6	**a** $m = 1$	**b**	$m < 1$	**c**	$m > 1$		
7	**a** $m = \pm 2\sqrt{2}$	**b**	$]-\infty, 2\sqrt{2}[\cup]\, 2\sqrt{2}, 0[$	**c**	$]-2\sqrt{2}, 2\sqrt{2}[$		
8	**a** $k = \pm 6\sqrt{2}$	**b**	$]-\infty, -6\sqrt{2}[\cup]\, 6\sqrt{2}, \infty[$	**c**	$]-6\sqrt{2}, 6\sqrt{2}[$		

Exercise 2.6.2

1 **a** Sum = –2 Product = 4 **b** Sum = 3 Product = –7 **c** Sum = 4 Product = –3

d Sum = $\frac{7}{5}$ Product = $\frac{3}{5}$ **e** Sum = $\frac{-5}{2}$ Product = $\frac{-3}{2}$

f Sum = $\frac{4}{9}$ Product = $\frac{-2}{9}$ **g** Sum = $\frac{7}{3}$ Product = $\frac{4}{3}$

h Sum = $\frac{-8}{3}$ Product = $\frac{-13}{5}$ **i** Sum = $\frac{3}{4}$ Product = $\frac{-1}{8}$

Consider the possibility of a zero denominator!

2 **a** 3 **b** –1 **c** $\frac{1}{2}$ **d** $\frac{-1}{2}$ **e** $\frac{-5}{3}$

f $\frac{9}{5}$ **g** $\frac{3}{2}$ **h** 4 **i** 7

3 The cubic case: $ax^3+bx^2+cx+d=0$ gives $x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$.

The factorized version is: $(x-\alpha)(x-\beta)(x-\gamma)=0$.

The only simple conclusion is that the product of the roots is $\alpha\beta\gamma=-\frac{d}{a}$.

4 This is related to the conjugate root theorem. The coefficients must be real.

Exercise 2.6.3

1 a 2 b –2 c ⅔ d 5 e 6

f –2½ g 2 h 1¼ i ⅓

2 a –6 b –⅔ c –3 d 1½ e ¼

f ¼ g $-{}^{1}/_{8}$ h $-{}^{11}/_{4}$ i –1¼

Exercise 2.7.1

4. Solve the following inequalities.

a $|4x + 2| \le 6$ **b** $|2x - 1| \le 5$ **c** $|4x - 2| \le 8$

d $|4x + 2| \le 0$ **e** $|x - 1| \le 8$ **f** $|3x + 3| \le 12$

g $\left|3 - \frac{x}{2}\right| \le 5$ **h** $\left|2 - \frac{x}{4}\right| \le 9$ **i** $\left|3x + \frac{1}{2}\right| \le \frac{3}{4}$

5. Solve the following inequalities.

a $|2x - 1| > 4$ b $|5 - 2x| > 2$ c $\left|1 - \frac{x}{2}\right| \ge 7$

d $\left|3 + \frac{1}{3}x\right| \ge 5$ e $3|6 - 4x| + 1 > 10$ f $12 - |4 - x| > 2$

g $\left|2 - \frac{x}{4}\right| > 9$ h $\left|3x + \frac{1}{2}\right| > \frac{3}{4}$ i $\left|3 - \frac{x}{2}\right| \ge 5$

6. For what value(s) of p does $\left|\frac{3x}{2} - 7\right| \le p - 3$ have no solutions?

7. Solve the following inequalities.

a $\frac{1}{2}x + 1 > |x|$ b $3 - x \ge |2x|$ c $|2x - 1| < x + 1$

8. Solve the following inequalities where $0 < a < 1$.

a $ax > |x - a|$ b $|x| < ax + 1$ c $\left|\frac{x}{a}\right| \ge x + a$

9. Find: **a** $\{x : 4 - |x| > |2x|\}$ **b** $\left\{x : \left|\frac{1}{3}x - 1\right| > |x|\right\}$.

Exercise 2.7.1

1 **a** $x < -4$ **b** $x \le -\frac{1}{5}$ **c** $x > 1$

d $x \le -6$ **e** $x > \frac{18}{7}$ **f** $x > \frac{3}{8}$

2 **a** $x > \frac{52}{11}$ **b** $x \le 1$ **c** $x \le \frac{10}{3}$

3 **a** $x < 1$ **b** $x < 2 - a$ **c** $x > \frac{2b}{3a}$ **d** $x \ge \frac{2}{(a+1)^2}$

4 **a** $-2 \le x \le 1$ **b** $-2 \le x \le 3$ **c** $-{}^{3}/_{2} \le x \le {}^{5}/_{2}$ **d** $x = -\frac{1}{2}$

e $-7 \le x \le 9$ **f** $-5 \le x \le 3$ **g** $-4 \le x \le 16$ **h** $-28 \le x \le 44$

i $-{}^{5}/_{12} \le x \le {}^{51}/_{12}$

5 **a** $x < -\frac{3}{2} \cup x > \frac{5}{2}$ **b** $x < \frac{3}{2} \cup x > \frac{7}{2}$ **c** $x \le -12 \cup x \ge 16$

d $x \le -24 \cup x \ge 6$ **e** $x < \frac{3}{4} \cup x > \frac{9}{4}$ **f** $-6 < x < 14$

g $x < -28 \cup x > 44$ **h** $x < -\frac{5}{12} \cup x > \frac{1}{12}$ **i** $x < -4 \cup x > 16$

6 $p < 3$

7 **a** $-\frac{2}{3} < x < 2$ **b** $-3 \le x \le 1$ **c** $0 < x < 2$

8 **a** $\frac{a}{1+a} < x < \frac{a}{1-a}$ **b** $\frac{-1}{1+a} < x < \frac{1}{1-a}$ **c** $\left]-\infty, \frac{-a^2}{a+1}\right] \cup \left[\frac{a^2}{a-1}, \infty\right[$

9 **a** $-\frac{4}{3} < x < \frac{4}{3}$ **b** $-\frac{3}{2} < x < \frac{3}{4}$

Exercise 2.7.2

1 **a** $]-\infty,-2[\cup]1,\infty[$ **b** $[-3,2]$ **c** $]-\infty,0]\cup[4,\infty[$

d $]\frac{1}{3},3[$ **e** $]-\infty,-1.5]\cup[-1,\infty[$ **f** $]0.75,2.5[$

2 **a** $]-\infty,-2[\cup]-1,\infty[$ **b** $]-2,3[$ **c** $]-\infty,-0.5]\cup[3,\infty[$

d $[-2,2]$ **e** $]\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}[$ **f** $]-\infty,-2]\cup[3,\infty[$

g $[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}]$ **h** $[-2.5,1]$ **i** $]-\infty,-3[\cup]0.5,\infty[$ **j** $]1,3[$

k $]-1,0.5[$ **l** $\varnothing$ **m** $\varnothing$

n $[-1.5,5]$ **o** $]-\infty,-2[\cup]\frac{1}{3},\infty[$

3 **a** $-1 < k < 0$ **b** $-2\sqrt{2} < k < 2\sqrt{2}$ **c** $n \le -0.5$

4 **a i** $]-\infty,-1[\cup]2,\infty[$ **ii** $[-1,2]$

b i $]-\infty,2[\cup]3,\infty[$ **ii** $[2,3]$

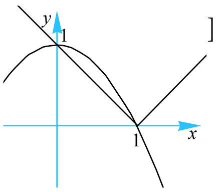
c i $]1,3[$ **ii** $]-\infty,1]\cup[3,\infty[$

d i $]-\frac{2}{3},1[$ **ii** $]-\infty,-\frac{2}{3}]\cup[1,\infty[$

e i $]-\infty,-2[\cup]2,\infty[$ **ii** $[-2,2]$

f i $]2-\sqrt{3}, 2+\sqrt{3}[$ **ii** $]-\infty,2-\sqrt{3}]\cup[2+\sqrt{3},\infty[$

5



$]0,1[$

6 $[-2,0.5]$

7 **a i** $]2-\sqrt{6}, 2-\sqrt{2}[\cup]2+\sqrt{2}, 2+\sqrt{6}[$

ii $]2(1-\sqrt{2}), 2(1+\sqrt{2}) \backslash\{2\}$

iii $]2(1-\sqrt{3}), 2(1+\sqrt{3})$

b i $[\frac{5-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}]$ **ii** all real values

8 **a** $\{x: x < -3\}$È$\{x: x > 2\}$

b $\{x: -1 < x < 4\}$

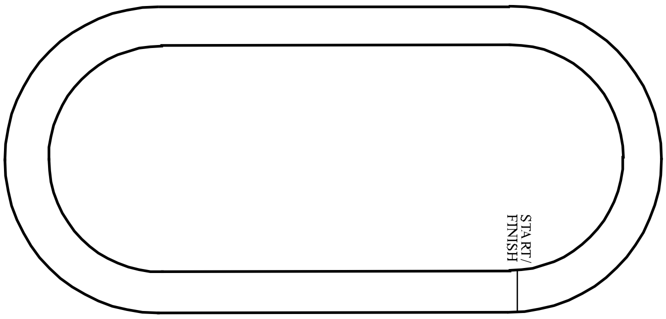
c i $\{x: x < 0.5\}$ **ii** $\{x: -2 < x < 0\}$

Exercise 3.1.1

1. Find the areas and perimeters of the following sectors.

	Radius	Angle
h	8.6 cm	$\frac{7\pi}{6}$
i	6.2 cm	$\frac{4\pi}{3}$
j	76 m	$\frac{11\pi}{6}$
k	12 cm	30°
l	14 m	60°
m	2.8 cm	120°
n	24.8 cm	270°
o	1.2 cm	15°

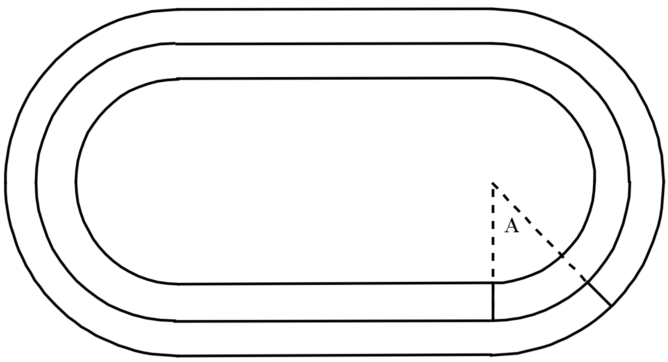
8.. The diagram shows a running track. The perimeter of the inside line is 400 metres and the length of each straight section is 100 metres.



a Find the radius of each of the semicircular parts of the inner track.

b If the width of the lane shown is 1 metre, find the perimeter of the outer boundary of the lane.

A second lane is added on the outside of the track. The starting positions of runners who have to run (anticlockwise) in the two lanes are shown.



c Find the value of angle A° (to the nearest degree) if both runners are to run 400 metres.

9. Find the angle subtended by at the centre of radius length 12 cm which forms a sector of area 80 sq. cm.

10. Find the angle subtended by an arc of a circle of radius length 10 cm which forms a sector of area 75 sq. cm.

11. A chord of length 32 cm is drawn in a circle of radius 20 cm.

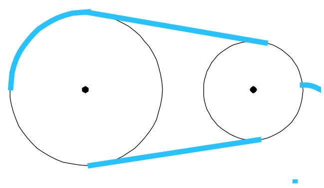
a Find the angle it subtends at the centre.

b Find: **i** the minor arc length **ii** the major arc length.

c Find the area of the minor sector.

12. Two circles of radii 6 cm and 8 cm have their centres 10 cm apart. Find the area common to both circles.

13.. Two pulleys of radii 16 cm and 20 cm have their centres 40 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if the string does not cross over.



14. Two pulleys of radii 7 cm and 11 cm have their centres 24 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if:

a the string cannot cross over.

b the string crosses over itself.

15. A sector of a circle has a radius of 15 cm and an angle of 216°. The sector is folded in such a way that it forms a cone, so that the two straight edges of the sector do not overlap.

a Find the base radius of the cone.

b Find the vertical height of the cone.

c Find the semi–vertical angle of the cone.

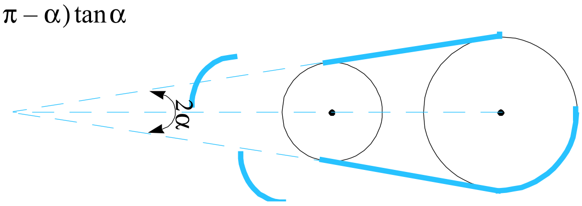
16. A taut belt passes over two discs of radii 4 cm and 12 cm as shown in the diagram.

a If the total length of the belt is 88 cm, show that $1 = (5.5 - \pi - \alpha)\tan\alpha$

b On the same set of axes, sketch the graphs of:

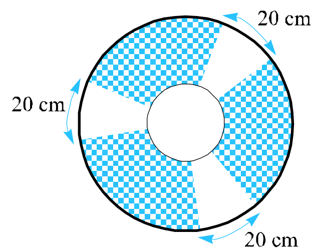
i $y = \dfrac{1}{\tan\alpha}$

ii $y = 5.5 - \pi - \alpha \cdot$



c Hence find $\{\alpha : 1 = (5.5 - \pi - \alpha)\tan\alpha\}$, giving your answer to two d.p.

17. The diagram shows a disc of radius 40 cm with parts of it painted. The smaller circle (having the same centre as the disc) has a radius of 10 cm. What area of the disc has not been painted in blue?



Exercise 3.1.1

1

a $\frac{169\pi}{150}$ cm², $5.2+\frac{13\pi}{15}$ cm

b $\frac{529\pi}{32}$ cm², $23+\frac{23\pi}{8}$ cm

c 242π cm², $88 + 11\pi$ cm

d $\frac{1156\pi}{75}$ m², $13.6+\frac{68\pi}{15}$ m

e $\frac{96\pi}{625}$ cm², $1.28+\frac{12\pi}{25}$ cm

f $\frac{361\pi}{15}$ cm², $15.2+\frac{19\pi}{3}$ cm

g 5248.8π m², $648 + 32.4\pi$ cm

h $\frac{12943\pi}{300}$ cm², $17.2+\frac{301\pi}{30}$ cm

i $\frac{1922\pi}{75}$ cm², $12.4+\frac{124\pi}{15}$ cm

j $\frac{15884\pi}{3}$ cm², $152+\frac{418\pi}{3}$ cm

k 12π cm², $24 + 2\pi$ cm

l $\frac{98\pi}{3}$ cm², $28+\frac{14\pi}{3}$ cm

m $\frac{196\pi}{75}$ cm², $5.6+\frac{28\pi}{15}$ cm

n $\frac{11532\pi}{25}$ cm², $49.6+\frac{186\pi}{5}$ cm

o $\frac{3\pi}{50}$ cm², $2.4+\frac{\pi}{10}$ cm

2 0.63^c, 36°

3 0.0942 m³

4 1.64^c

5 79 cm

6 5.25 cm²

7 $\frac{\sqrt{50}\pi}{5}^c$

8 **a** 31.83 m **b** 406.28 m **c** 11°

9 1.11^c

10 0.75^c

11 **a** 1.85^c **b i** 37.09 cm **ii** 88.57 cm **c** 370.92 cm²

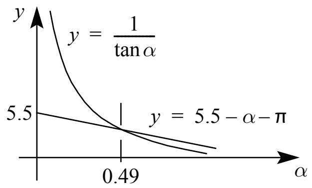
12 26.57 cm²

13 193.5 cm

14 **a** 105.22 cm **b** 118.83 cm

15 **a** 9 cm **b** 12 cm **c** 36° 52’

16 **b**



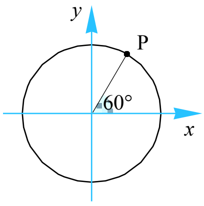
c 0.49

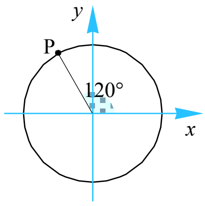
17 1439.16 cm^2

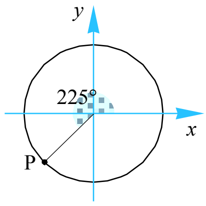
Exercise 3.2.1

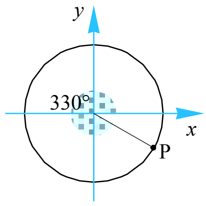
7. Find the coordinates of the point P on the following unit circles.

a **b** **c** **d**









8. Find the exact value of:

a $\sin\frac{11\pi}{6}\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6}\cos\frac{11\pi}{6}$ **b** $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

c $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ **d** $\cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

9. Show that the following relationships are true.

a $\sin 2\theta = 2\sin\theta\cos\theta$, where $\theta = \frac{\pi}{3}$ **b** $\cos 2\theta = 2\cos^2\theta - 1$, where $\theta = \frac{\pi}{6}$

c $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, where $\theta = \frac{2\pi}{3}$ **d** $\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$, where $\theta = \frac{2\pi}{3}$ and $\phi = -\frac{\pi}{3}$

10. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\sin(\pi + \theta)$ **b** $\sin(2\pi - \theta)$ **c** $\cos\left(\frac{\pi}{2} + \theta\right)$

11. Given that $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos(\pi - \theta)$ **b** $\sec\theta$ **c** $\sin\left(\frac{\pi}{2} - \theta\right)$

12. Given that $\tan\theta = k$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\tan(\pi + \theta)$ **b** $\tan\left(\frac{\pi}{2} + \theta\right)$ **c** $\tan(-\theta)$

13. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos\theta$ **b** $\sec\theta$ **c** $\cos(\pi+\theta)$

14. Given that $\cos\theta = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi+\theta)$

15. Given that $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\sin\theta$ **b** $\tan\left(\frac{\pi}{2}+\theta\right)$ **c** $\sec\theta$

16. Given that $\cos\theta = k$ and $\frac{3\pi}{2} < \theta < 2\pi$, find:

a $\cos(\pi-\theta)$ **b** $\sin\theta$ **c** $\cot\theta$

17. Given that $\sin\theta = -k$ and $\pi < \theta < \frac{3\pi}{2}$,

find:

a $\cos\theta$ **b** $\tan\theta$ **c** $\mathrm{cosec}\left(\frac{\pi}{2}+\theta\right)$

18. Simplify the following.

a $\dfrac{\sin(\pi-\theta)\cos\left(\frac{\pi}{2}+\theta\right)}{\sin(\pi+\theta)}$ **b** $\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}{\sin^2\theta}$ **c** $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\theta}$

d $\tan(\pi+\theta)\cot\theta$ **e** $\cos(2\pi-\theta)\mathrm{cosec}\theta$ **f** $\dfrac{\sec\theta}{\mathrm{cosec}\theta}$

19. If $0 \le \theta \le 2\pi$, find all values of x such that:

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\cos x = \frac{1}{2}$ **c** $\tan x = \sqrt{3}$

d $\cos x = -\frac{\sqrt{3}}{2}$ **e** $\tan x = -\frac{1}{\sqrt{3}}$ **f** $\sin x = -\frac{1}{2}$

Exercise 3.2.2

6. Prove $\sin^2 x(1 + n\cot^2 x) + \cos^2 x(1 + n\tan^2 x) = \sin^2 x(n + \cot^2 x) + \cos^2 x(n + \tan^2 x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \dfrac{mk}{m^2 - k^2}$.

8. If $x = k\sec^2\phi + m\tan^2\phi$ and $y = l\sec^2\phi + n\tan^2\phi$, prove that $\dfrac{x-k}{k+m} = \dfrac{y-l}{l+n}$.

9. Given that $\tan\theta = \dfrac{2a}{a^2-1}$, $0 < \theta < \dfrac{\pi}{2}$, find: **a** $\sin\theta$ **b** $\cos\theta$

10. **a** If $\sin x + \cos x = 1$, find the values of: **i** $\sin^3 x + \cos^3 x$ **ii** $\sin^4 x + \cos^4 x$

b Hence, deduce the value of $\sin^k x + \cos^k x$, where k is a positive integer.

11. If $\tan\phi = -\dfrac{1}{\sqrt{x^2-1}}$, $\dfrac{\pi}{2} < \phi < \pi$, find, in terms of x,

a $\sin\phi + \cos\phi$ b $\sin\phi - \cos\phi$ c $\sin^4\phi - \cos^4\phi$

12. Find: **a** the maximum value of **b** the minimum value of

i $\cos^2\theta + 5$ **ii** $\dfrac{5}{3\sin^2\theta + 2}$ **iii** $2\cos^2\theta + \sin\theta - 1$

13. **a** Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find b.

b Hence, find all values of ϕ that satisfy the relationship described in part **a**.

14. Find: **a** the maximum value of **b** the minimum value of

i $5^{3\sin\theta - 1}$ **ii** $3^{1 - 2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find: **a** $(\sin\theta + \cos\theta)^2$, $\sin\theta + \cos\theta > 0$.

b $\sin^3\theta + \cos^3\theta$, $\sin\theta + \cos\theta > 0$ b

16. **a** Given that $\sin\phi = \frac{1-a}{1+a}, 0 < \phi < \frac{\pi}{2}$, find $\tan\phi$.

b Given that $\sin\phi = 1 - a, \frac{\pi}{2} < \phi < \pi$, find : **i** $2 - \cos\phi$ **ii** $\cot\phi$

17. Find:

a the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec} x - \tan x)$, $0 < x < \pi$.

b the values of $\sin x$, where $3\cos x = 2 + \frac{1}{\cos x}, 0 \le x \le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x, 0 \le x \le \pi$.

Exercise 3.2.1

1	**a**	120°	**b**	108°	**c**	216°	**d**	50°
2	**a**	π^c	**b**	$\frac{3\pi^c}{2}$	**c**	$\frac{7\pi^c}{9}$	**d**	$\frac{16\pi^c}{9}$
3	**a**	$\frac{\sqrt{3}}{2}$	**b**	$-\frac{1}{2}$	**c**	$-\sqrt{3}$	**d**	-2
	e	$-\frac{1}{2}$	**f**	$-\frac{\sqrt{3}}{2}$	**g**	$\frac{1}{\sqrt{3}}$	**h**	$\sqrt{3}$
	i	$-\frac{1}{\sqrt{2}}$	**j**	$-\frac{1}{\sqrt{2}}$	**k**	1	**l**	$-\sqrt{2}$
	m	$-\frac{1}{\sqrt{2}}$	**n**	$\frac{1}{\sqrt{2}}$	**o**	-1	**p**	$\sqrt{2}$
	q	0	**r**	1	**s**	0	**t**	undefined
4	**a**	0	**b**	-1	**c**	0	**d**	-1
	e	$\frac{1}{\sqrt{2}}$	**f**	$-\frac{1}{\sqrt{2}}$	**g**	-1	**h**	$\sqrt{2}$
	i	$-\frac{1}{2}$	**j**	$-\frac{\sqrt{3}}{2}$	**k**	$\frac{1}{\sqrt{3}}$	**l**	$\sqrt{3}$
	m	$-\frac{\sqrt{3}}{2}$	**n**	$\frac{1}{2}$				
5	**a**	$\frac{1}{2}$	**b**	$\frac{\sqrt{3}}{2}$	**c**	11	**d**	$\frac{1}{2}$
	e	$-\frac{1}{\sqrt{3}}$	**f**	$-\frac{1}{2}$	**g**	$-\sqrt{2}$		
6	**a**	$-\frac{1}{2}$	**b**	$-\frac{1}{\sqrt{2}}$	**c**	$\sqrt{3}$	**d**	-2
	e	1	**f**	$\frac{1}{2}$	**g**	$-\frac{1}{\sqrt{3}}$	**h**	$-\frac{\sqrt{3}}{2}$
	i	$-\frac{2}{\sqrt{3}}$	**j**	$\frac{1}{\sqrt{3}}$	**k**	$\frac{2}{\sqrt{3}}$		

7	**a**	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**b**	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**c**	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	**d**	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
8	**a**	0	**b**	$\frac{\sqrt{3}}{2}$	**c**	$\frac{1}{\sqrt{3}}$	**d**	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
10	**a**	$-\frac{2}{3}$	**b**	$-\frac{2}{3}$	**c**	$-\frac{2}{3}$		
11	**a**	$-\frac{2}{5}$	**b**	$\frac{5}{2}$	**c**	$\frac{2}{5}$		
12	**a**	k	**b**	$-\frac{1}{k}$	**c**	$-k$		
13	**a**	$\frac{\sqrt{5}}{3}$	**b**	$\frac{3}{\sqrt{5}}$	**c**	$-\frac{\sqrt{5}}{3}$		
14	**a**	$-\frac{3}{5}$	**b**	$\frac{3}{4}$	**c**	$\frac{4}{5}$		
15	**a**	$\frac{4}{5}$	**b**	$\frac{3}{4}$	**c**	$-\frac{5}{3}$		
16	**a**	$-k$	**b**	$-\sqrt{1-k^2}$	**c**	$-\frac{k}{\sqrt{1-k^2}}$		
17	**a**	$-\sqrt{1-k^2}$	**b**	$\frac{k}{\sqrt{1-k^2}}$	**c**	$-\frac{1}{\sqrt{1-k^2}}$		
18	**a**	$\sin\theta$	**b**	$\cot\theta$	**c**	1	**d**	1
	e	$\cot\theta$	**f**	$\tan\theta$				
19	**a**	$\frac{\pi}{3}, \frac{2\pi}{3}$	**b**	$\frac{\pi}{3}, \frac{5\pi}{3}$	**c**	$\frac{\pi}{3}, \frac{4\pi}{3}$	**d**	$\frac{5\pi}{6}, \frac{7\pi}{6}$
	e	$\frac{5\pi}{6}, \frac{11\pi}{6}$	**f**	$\frac{7\pi}{6}, \frac{11\pi}{6}$				

Exercise 3.2.2

3 **a** $x^2 + y^2 = k^2, -k \le x \le k$ **b** $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, -b \le x \le b$

c $(x-1)^2 + (2-y)^2 = 1, 0 \le x \le 2$ **d** $\frac{(1-x)^2}{b^2} + \frac{(y-2)^2}{a^2} = 1$

e $5x^2 + 5y^2 + 6xy = 16$

4 **a i** $-\frac{4}{5}$ **ii** $-\frac{5}{3}$ **b i** $\frac{4}{\sqrt{7}}$ **ii** $-\frac{\sqrt{7}}{3}$

5 **a** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$ **d** $\frac{\pi}{2}, \frac{3\pi}{2}$

9 **a** $\frac{2a}{a^2+1}$ **b** $\frac{a^2-1}{a^2+1}$

10 **a i** 1 **ii** 1 **b** 1

11 **a** $\frac{1-\sqrt{x^2-1}}{x}$ **b** $\frac{1+\sqrt{x^2-1}}{x}$ **c** $\frac{2}{x^2} - 1$

12 **a i** 6 **ii** $\frac{5}{2}$ **iii** $\frac{9}{8}$

b i 5 **ii** 1 **iii** −2

13 **a** ±2 **b** $\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$ or $\frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$

14 **a i** 25 **ii** $\frac{1}{5^4}$ **b i** 27 **ii** $\frac{1}{3}$

15 **a** $1 + 2k$ **b** $(1-k)\sqrt{1+2k}$

16 **a** $\frac{1-a}{2\sqrt{a}}$ **b i** $2 + \sqrt{2a - a^2}$ **ii** $\frac{-\sqrt{2a-a^2}}{1-a}$

17 **a** $\frac{2}{3}$ **b** $0, \pm\frac{2\sqrt{2}}{3}$

18 $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

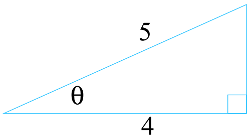
Extra Examples

Example 3.3.8

If $\sin\theta = \frac{3}{5}$ and $\cos\phi = -\frac{12}{13}$, where $0 \le \theta \le \frac{\pi}{2}$ and $\pi \le \phi \le \frac{3\pi}{2}$, find:

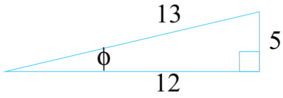
a $\sin(\theta + \phi)$ b $\cos(\theta + \phi)$ c $\tan(\theta - \phi)$

We start by drawing two right-angled triangles satisfying the given conditions:



$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\sin\phi = \frac{5}{13}$

$\tan\phi = \frac{5}{12}$

$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$

However, we cannot simply substitute the above ratios into this expression as we now need to consider the sign of the ratios.

As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\sin(\theta + \phi) = \frac{3}{5} \times -\frac{12}{13} + -\frac{5}{13} \times \frac{4}{5} = -\frac{56}{65}$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

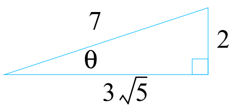
As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\cos(\theta + \phi) = \frac{4}{5} \times -\frac{12}{13} - \frac{3}{5} \times -\frac{5}{13} = -\frac{33}{65}$

Example 3.3.9

If $\sin\theta = \frac{2}{7}$, where $\frac{\pi}{2} \le \theta \le \pi$, find: a $\sin 2\theta$ b $\cos 2\theta$ c $\tan 2\theta$

We start by drawing the relevant right-angled triangle:



a $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{2}{7} \times -\frac{3\sqrt{5}}{7}$

$= -\frac{12\sqrt{5}}{49}$

b $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2 \times \left(\frac{2}{7}\right)^2 = \frac{41}{49}$

c $\quad \tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{-\frac{12\sqrt{5}}{49}}{\frac{41}{49}} = -\dfrac{12\sqrt{5}}{41}$

Example 3.3.10

Prove that: a $\quad \sin 2\alpha \tan\alpha + \cos 2\alpha = 1$ b $\quad 2\cot 2\beta = \cot\beta - \tan\beta$

a $\quad$ L.H.S $= \sin 2\alpha\tan\alpha + \cos 2\alpha = 2\sin\alpha\cos\alpha \times \dfrac{\sin\alpha}{\cos\alpha} + (1 - 2\sin^2\alpha)$

$$= 2\sin^2\alpha + 1 - 2\sin^2\alpha \qquad = \text{R.H.S}$$
$$= 1$$

b $\quad$ R.H.S $= \cot\beta - \tan\beta = \dfrac{\cos\beta}{\sin\beta} - \dfrac{\sin\beta}{\cos\beta}$

$$= \frac{\cos^2\beta - \sin^2\beta}{\sin\beta\cos\beta}$$
$$= \frac{\cos 2\beta}{\frac{1}{2}\sin 2\beta}$$
$$= 2\frac{\cos 2\beta}{\sin 2\beta}$$
$$= 2\cot 2\beta$$

$= \text{L.H.S}$

Notice that, when proving identities, when all else fails, then express everything in terms of sine and cosine. This will always lead to the desired result – even though sometimes the working seems like it will only grow and grow – eventually, it does simplify. Be persistent.

To prove a given identity, any one of the following approaches can be used:

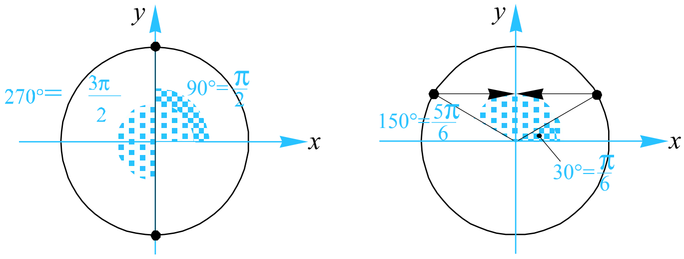
1. Start with the L.H.S and then show that L.H.S = R.H.S
2. Start with the R.H.S and then show that R.H.S = L.H.S
3. Show that L.H.S = p, show that R.H.S = p $\Rightarrow$ L.H.S = R.H.S
4. Start with L.H.S = R.H.S $\Rightarrow$ L.H.S – R.H.S = 0 .

When using approaches 1 and 2, choose whichever side has more to work with.

Example 3.3.11

Find all values of x, such that $\sin 2x = \cos x$, where $0 \le x \le 2\pi$.

$\sin 2x = \cos x \Leftrightarrow 2\sin x\cos x = \cos x$



$$\Leftrightarrow 2\sin x\cos x - \cos x = 0$$
$$\Leftrightarrow \cos x(2\sin x - 1) = 0$$
$$\Leftrightarrow \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Now, $\cos x = 0,\ 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

and

$\sin x = \frac{1}{2},\ 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Example 3.3.12

Simplify $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$.

Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are real numbers.

Hence find the maximum value of $\cos\theta - \sin\theta$.

$$\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\sin\theta \times \frac{1}{\sqrt{2}} + \cos\theta \times \frac{1}{\sqrt{2}}\right]$$
$$= \sin\theta + \cos\theta$$

In this instance, as the statement needs to be true for all values of θ, we will determine the values of R and α by setting $R\cos(\theta + \alpha) \equiv \cos\theta - \sin\theta$.

Now, $R\cos(\theta + \alpha) = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Therefore, we have that $R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \equiv \cos\theta - \sin\theta$

$$\Rightarrow R\cos\theta\cos\alpha = \cos\theta \Leftrightarrow R\cos\alpha = 1 \quad \text{– (1)}$$

$$\Rightarrow R\sin\theta\sin\alpha = \sin\theta \Leftrightarrow R\sin\alpha = 1 \quad \text{– (2)}$$

Dividing (2) by (1) we have $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1} \Leftrightarrow \tan\alpha = 1 \therefore \alpha = \frac{\pi}{4}$

Substituting into (1) we have $R\cos\frac{\pi}{4} = 1 \Leftrightarrow R \times \frac{1}{\sqrt{2}} = 1 \therefore R = \sqrt{2}$.

Therefore, $\cos\theta - \sin\theta \equiv \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

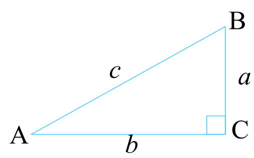
Then, as the maximum value of the cosine is 1, the maximum of $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$ is $\sqrt{2}$.

Exercise 3.3.1

10. Prove that: **a** $\frac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\frac{1}{2}x$ **b** $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

c $\sin^4\phi = \frac{3}{8} + \frac{1}{8}\cos 4\phi - \frac{1}{2}\cos 2\phi$ **d** $\sin x = \frac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$

11. For the right-angled triangle shown, prove that:



a $\sin 2\alpha = \frac{2ab}{c^2}$ **b** $\cos 2\alpha = \frac{b^2 - a^2}{c^2}$

c $\sin\frac{1}{2}\alpha = \sqrt{\frac{c-b}{2c}}$ **d** $\cos\frac{1}{2}\alpha = \sqrt{\frac{c+b}{2c}}$

12. Find the exact value $\tan\frac{\pi}{8}$.

13. Given that $\alpha + \beta + \gamma = \pi$, prove that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

14. Solve the following for $0 \le x \le 2\pi$.

a $\sin x = \sin 2x$ **b** $\sin x = \cos 2x$ **c** $\tan 2x = 4\tan x$

15. **a** Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta + \alpha)$, express R and α in terms of a and b.

b Find the maximum value of $5 + 4\sin\theta + 3\cos\theta$.

16. **a** Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, express R and α in terms of a and b.

b Find the minimum value of $2 + 12\cos\theta + 5\sin\theta$.

17. Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}x\right) = \sec x + \tan x$.

18. Show that if $t = \tan\frac{\pi}{12}$ then $t^2 + 2\sqrt{3}t = 1$. Hence find the exact value of $\tan\frac{\pi}{12}$.

Exercise 3.3.1

1

a $\sin\alpha\cos\phi + \cos\alpha\sin\phi$ **b** $\cos 3\alpha\cos 2\beta - \sin 3\alpha\sin 2\beta$ **c** $\sin 2x\cos y - \cos 2x\sin y$

d $\cos\phi\cos 2\alpha + \sin\phi\sin 2\alpha$ **e** $\frac{\tan 2\theta - \tan\alpha}{1 + \tan 2\theta\tan\alpha}$ **f** $\frac{\tan\phi - \tan 3\omega}{1 + \tan\phi\tan 3\omega}$

2

a $\sin(2\alpha - 3\beta)$ **b** $\cos(2\alpha + 5\beta)$ **c** $\sin(x + 2y)$

d $\cos(x - 3y)$ **e** $\tan(2\alpha - \beta)$ **f** $\tan x$

g $\tan\left(\frac{\pi}{4} - \phi\right)$ **h** $\sin\left(\frac{\pi}{4} + \alpha + \beta\right)$ **i** $\sin 2x$

3 **a** $-\frac{56}{65}$ **b** $\frac{33}{65}$ **c** $-\frac{16}{63}$

4 **a** $\frac{16}{65}$ **b** $\frac{63}{65}$ **c** $\frac{56}{33}$

5 **a** $-\frac{5\sqrt{11}}{18}$ **b** $-\frac{7}{18}$ **c** $\frac{5\sqrt{11}}{7}$ **d** $\frac{35\sqrt{11}}{162}$

6 **a** $-\frac{3}{5}$ **b** $-\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{24}{7}$

7 **a** $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **b** $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **c** $-\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **d** $\sqrt{3} - 2$

8 **a** $\frac{2ab}{a^2 + b^2}$ **b** $\frac{a^2 + b^2}{2ab}$ **c** $\frac{a^4 - 6a^2b^2 + b^4}{(a^2 + b^2)^2}$ **d** $\frac{2ab}{b^2 - a^2}$

12 $\sqrt{2} - 1$

14 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **c** $0, \pi, 2\pi, \alpha, \pi \pm \alpha, 2\pi - \alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15 **a** $R = \sqrt{a^2 + b^2}, \tan\alpha = \frac{b}{a}$ **b** 10

16 **a** $R = \sqrt{a^2 + b^2}, \tan\alpha = \frac{b}{a}$ **b** -11

18 $2 - \sqrt{3}$

Exercise 3.4.1

7. Sketch graphs of the following functions for x-values in the interval $[-2\pi, 2\pi]$.

a $y = \sin(2x)$

b $y = -\cos\left(\frac{x}{2}\right)$

c $y = 3\tan\left(x - \frac{\pi}{4}\right)$

d $y = 2\sin\left(x - \frac{\pi}{2}\right)$

e $y = 1 - 2\sin(2x)$

f $y = -2\cos\left(\frac{x-\pi}{2}\right)$

g $y = 3\tan\left[\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right] - 3$

h $y = 3\cos\left(x + \frac{\pi}{4}\right)$

i $y = 2\sin\left[\frac{1}{3}\left(x + \frac{2\pi}{3}\right)\right] - 1$

j $y = 3\tan(2x + \pi)$

k $y = 4\sin\left(\frac{x + \frac{\pi}{2}}{2}\right)$

l $y = 2 - \sin\left(\frac{2(x-\pi)}{3}\right)$

m $y = 2\cos(\pi x)$

n $y = 2\sin[\pi(x+1)]$

8.

a i Sketch one cycle of the graph of the function $f(x) = \sin x$.

ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii Hence, sketch one cycle of the graph of the function $g(x) = \operatorname{cosec} x$.

bi Sketch one cycle of the graph of the function $f(x) = \cos x$.

ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii Hence, sketch one cycle of the graph of the function $g(x) = \sec x$.

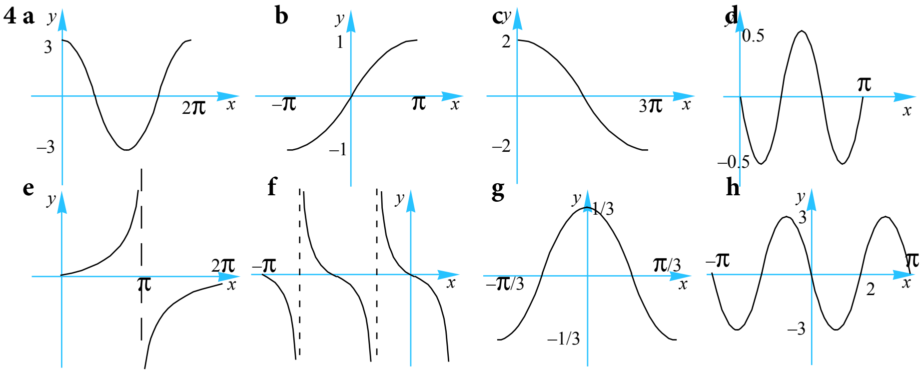
ci Sketch one cycle of the graph of the function $f(x) = \tan x$.

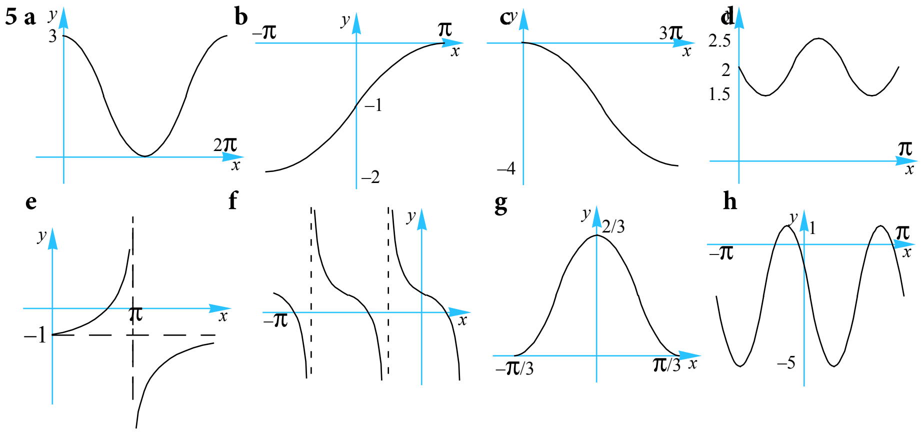
ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

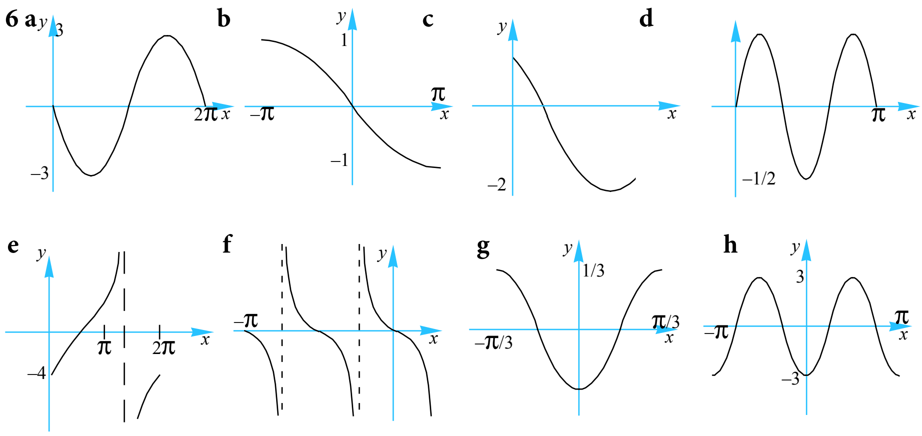
iii Hence, sketch one cycle of graph of the function $g(x) = \cot x$.

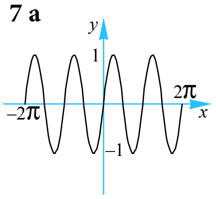
Exercise 3.4.1

1	**a**	4π	**b**	$\frac{2\pi}{3}$	**c**	3π	**d**	4π	**e**	2	**f**	$\frac{\pi}{2}$
2	**a**	5	**b**	3	**c**	5	**d**	0.5				
3	**a**	2π, 2	**b**	6π, 3	**c**	π	**d**	π	**e**	π, 4		
	f	π, 3	**g**	6π	**h**	$\frac{2\pi}{3}, \frac{1}{4}$	**i**	3π	**j**	$\frac{8\pi}{3}, \frac{2}{3}$		







7 a


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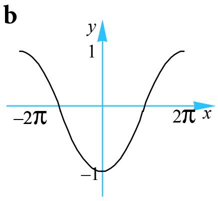

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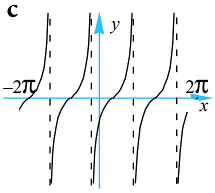

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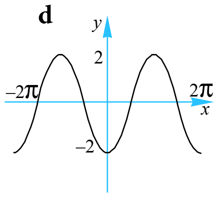

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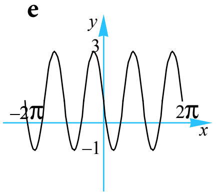

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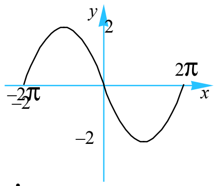

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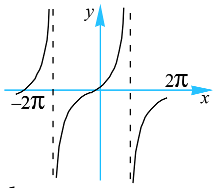
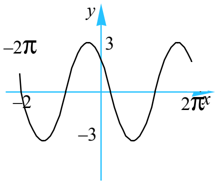
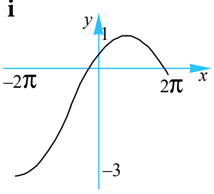


h


i


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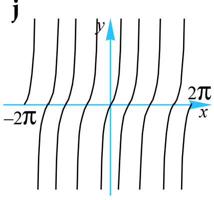

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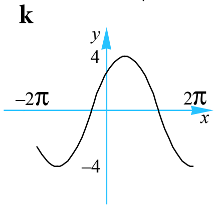

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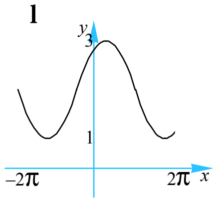

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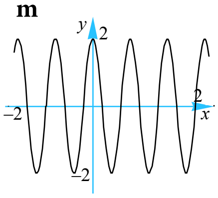

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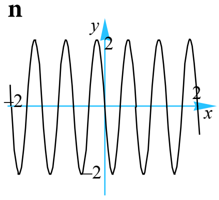
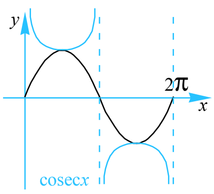
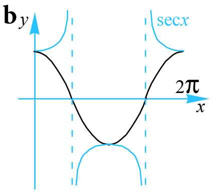
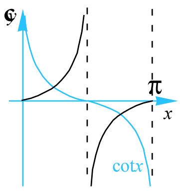


8 a


b


c


Example 3.5.6

Give the exact value of: a $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$ b $\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

a as $\frac{1}{3} \in [-1, 1]$ and $\frac{3}{4} \in [-1, 1]$ then both $\text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $\text{Sin}^{-1}\left(\frac{3}{4}\right)$ exist.

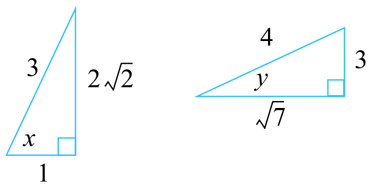
Using the compound angle formula for sine, we have that

$$\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$$

$$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)+\sin\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)\cos\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)$$

$$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)+\frac{3}{4}\times\frac{1}{3}$$

However, we now need to construct two right-angled triangles to evaluate the first part of the expression. Let $x = \text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $y = \text{Sin}^{-1}\left(\frac{3}{4}\right)$ so that we have the following triangles:



Meaning that $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right) = \sin x = \frac{2\sqrt{2}}{3}$

and $\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \cos y = \frac{\sqrt{7}}{4}$

Therefore, $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \frac{2\sqrt{2}}{3}\times\frac{\sqrt{7}}{4}+\frac{3}{4}\times\frac{1}{3} = \frac{2\sqrt{14}+3}{12}$

As $\frac{2}{\sqrt{5}} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$ exists. Next, let $\theta = \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$, i.e. $\text{Sin}\theta = \frac{2}{\sqrt{5}}$.

Then, using the double-angle formula for cosine, we have

$$\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \cos(2\theta) = 1-2\sin^2\theta$$

$$= 1-2[\sin\theta]^2$$

$$= 1-2\left[\frac{2}{\sqrt{5}}\right]^2$$

$$= -\frac{3}{5}$$

Exercise 3.5.1

6. Find the exact value of:

a $\sin\left[\mathrm{Sin}^{-1}\left(\frac{3}{5}\right)+\mathrm{Sin}^{-1}\left(\frac{4}{5}\right)\right]$ b $\sin\left[\mathrm{Sin}^{-1}\left(\frac{3}{5}\right)+\mathrm{Sin}^{-1}\left(-\frac{4}{5}\right)\right]$

c $\cos\left[\mathrm{Tan}^{-1}\left(\frac{4}{3}\right)-\mathrm{Cos}^{-1}\left(\frac{5}{13}\right)\right]$ d $\tan\left[\mathrm{Tan}^{-1}\left(\frac{4}{3}\right)+\mathrm{Tan}^{-1}\left(\frac{3}{4}\right)\right]$

e $\sin\left(2\mathrm{Arcsin}\left(\frac{2}{3}\right)\right)$ f $\cos\left(2\mathrm{Tan}^{-1}\left(-\frac{1}{2}\right)\right)$

g $\tan\left(2\mathrm{Cos}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ h $\cos\left(2\mathrm{Sin}^{-1}\left(-\frac{1}{2}\right)\right)$

7. Prove that: a $\mathrm{Sin}^{-1}\left(\frac{7}{25}\right) = \mathrm{Cos}^{-1}\left(\frac{24}{25}\right)$ b $\mathrm{Tan}^{-1}\left(\frac{1}{2}\right)+\mathrm{Tan}^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

c $\mathrm{Tan}^{-1}(1)+\mathrm{Tan}^{-1}(2)+\mathrm{Tan}^{-1}(3) = \pi$

8. Prove that: a $\mathrm{Sin}^{-1}x = \mathrm{Cos}^{-1}\sqrt{1-x^2},\ 0 \le x \le 1$

b $\mathrm{Cos}^{-1}x = \mathrm{Sin}^{-1}\sqrt{1-x^2},\ 0 \le x \le 1$

c $\mathrm{Tan}^{-1}x+\mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for all real x and y, xy ¦ 1.

d $\mathrm{Tan}^{-1}x-\mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$, for all real x and y, xy ¦ -1.

9. Find: a $\tan(\mathrm{Cos}^{-1}k)$, where $-1 \le k \le 1, k \ne 0$.

b $\cos(\mathrm{Tan}^{-1}k)$, where k is a real number.

10. State the implied domain of the following functions and sketch their graphs.

a $f(x) = \mathrm{Sin}^{-1}\left(\frac{x}{2}\right)+\frac{\pi}{2}$ b $f(x) = \mathrm{Cos}^{-1}(2x)-\pi$

c $g(x) = 2\mathrm{Sin}^{-1}(x-1)$ d $h(x) = \mathrm{Cos}^{-1}(x+2)-\frac{\pi}{2}$

11. a Prove that $\mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \mathrm{Tan}^{-1}x$, for all real x.

b Prove that $\mathrm{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \mathrm{Tan}^{-1}x$, for all real x.

12. a On the same set of axes sketch the graphs of $y = \mathrm{Cos}^{-1}x$ and $y = \mathrm{Sin}^{-1}x$.

b Hence, deduce the value of k, where

i $\mathrm{Cos}^{-1}x+\mathrm{Sin}^{-1}x = k, -1 \le x \le 0$

ii $\mathrm{Cos}^{-1}x+\mathrm{Sin}^{-1}x = k, 0 \le x \le 1$

On a separate set of axes, sketch the graph of $y = \mathrm{Cos}^{-1}x+\mathrm{Sin}^{-1}x, -1 \le x \le 1$.

13. Prove that if $n > 1$ then $\text{Arctan}\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n+1}\right) = \text{Arctan}\left(\frac{1}{1+n(n+1)}\right)$.

Hence, find $\sum_{i=1}^{n} \text{Arctan}\left(\frac{1}{1+i(i+1)}\right)$.

Exercise 3.5.1

1 **a** $\frac{\pi}{4}$ **b** $\frac{\pi}{2}$ **c** π **d** $\frac{\pi}{3}$

e $\frac{\pi}{4}$ **f** $-\frac{\pi}{3}$ **g** 1.1071^c **h** -0.7754^c

i 0.0997^c **j** 1.2661^c **k** -0.6435^c **l** 1.3734^c

m undefined **n** -1.5375^c **o** 1.0654^c

2 **a** -1 **b** $\frac{\sqrt{3}}{4}$ **c** $-\frac{1}{3\sqrt{2}}$

4 $\frac{1}{3}, \frac{1}{2}$

5 **a** $\frac{2}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$

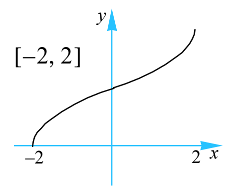
d $\frac{3}{4}$ **e** $\frac{3\sqrt{2}}{4}$ **f** -1

6 **a** 1 **b** $-\frac{7}{25}$ **c** $\frac{63}{65}$ **d** undefined

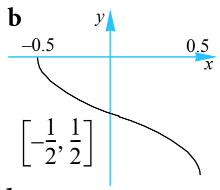
e $\frac{4\sqrt{5}}{9}$ **f** $\frac{3}{5}$ **g** $\frac{4}{3}$ **h** $\frac{1}{2}$

9 **a** $\frac{\sqrt{1-k^2}}{k}$ **b** $\frac{1}{\sqrt{1+k^2}}$

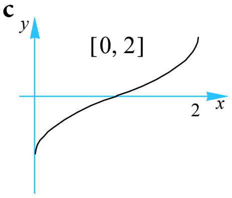
10 a



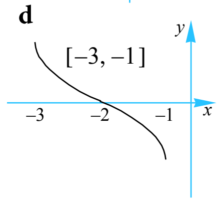
b



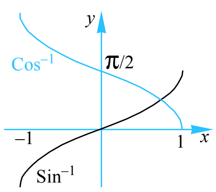
c



d

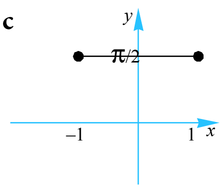


12 a



b i $\frac{\pi}{2}$ [redacted] **ii** $\frac{\pi}{2}$

c



13 $\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{n+1}\right)$

Exercise 3.6.1

6. Solve the following equations for the intervals indicated, giving exact answers:

e $\cos^2 x = 2\cos x,\ -\pi \le x \le \pi$

f $\sec 2x = \sqrt{2},\ 0 \le x \le 2\pi$

g $2\sin^2 x - 3\cos x = 2,\ 0 \le x \le 2\pi$

h $\sin 2x = 3\cos x,\ 0 \le x \le 2\pi$

7. Find:

a $3\tan^2 x + \tan x = 2,\ 0 \le x \le 2\pi$.

b $\tan^3 x + \tan^2 x = 3\tan x + 3,\ 0 \le x \le 2\pi$.

8. If $0 \le x \le 2\pi$, find:

a $\sin^2 2x - \frac{1}{4} = 0$ b $\tan^2\left(\frac{x}{2}\right) - 3 = 0$ c $\cos^2(\pi x) = 1$

9. If $0 \le x \le 2\pi$, find:

a $\sec^2 x + 2\sec x = 8$ b $\sec^2 x = 2\tan x + 4$

c $\cot^2 x - \sqrt{3}\cot x = 0$ d $6\operatorname{cosec}^2 x = 8 + \cot x$

10. a Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$.

b Solve $\sqrt{3}\sin x + \cos x = 1,\ 0 \le x \le 2\pi$.

11. a Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$.

b Solve $\sin x - \sqrt{3}\cos x = -1,\ 0 \le x \le 2\pi$.

12. Find x if $2\sin\left(x + \frac{\pi}{3}\right) + 2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{3},\ 0 \le x \le 2\pi$.

13. a Sketch the graph of $f(x) = \sin x,\ 0 \le x \le 4\pi$.

b Hence, find: i $\left\{x \mid \sin x > \frac{1}{2}\right\} \cap \{x \mid 0 < x < 4\pi\}$.

ii $\{x \mid \sqrt{3}\sin x < -1\} \cap \{x \mid 0 < x < 4\pi\}$.

14. a i On the same set of axes sketch the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ for $0 \le x \le 2\pi$.

ii Find $\{x | \sin x < \cos x, 0 \le x \le 2\pi\}$.

b i On the same set of axes sketch the graphs of $f(x) = \sin 2x$ and $g(x) = \cos x$ for $0 \le x \le 2\pi$.

ii Find $\{x | \sin 2x < \cos x, 0 \le x \le 2\pi\}$.

15. Show that $\{x : \sqrt{3}\cos x - \sin x = 1, x \in \mathbb{R}\} = \left\{x : x = 2n\pi + \frac{\pi}{6}\right\} \cup \left\{x : x = 2n\pi - \frac{\pi}{2}\right\}$, where n is an integer.

16. Find: ai $\left\{x : \sin x = \sin\alpha, x \in \mathbb{R}, 0 < \alpha < \frac{\pi}{2}\right\}$.

ii $\left\{x : \sin x \ge \sin\alpha, x \in \mathbb{R}, 0 < \alpha < \frac{\pi}{2}\right\}$

b $\{x : \sin 3x = \sin 2x, x \in \mathbb{R}\}$.

c $\{x : \cos 3x = \sin 2x, x \in \mathbb{R}\}$.

17. Given that the quadratic equation $x^2 - \sqrt{8}\cos\theta x + 3\cos\theta = 1$ has equal roots, find:

a $\theta, 0 \le \theta \le 2\pi$ b the roots of the quadratic.

18. a Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b Using the substitution $t = 2\cos\theta$, show that $t^3 = 3t + 1$ becomes $\cos 3\theta = \frac{1}{2}$.

c Hence find the exact values of the roots of the equation $x^3 - 3x - 1 = 0$.

19. Find $\{x | \cos x + \cos 2x + \cos 3x = 0, -\pi < x < \pi\}$

20. a Given that $2\sin 2x + \cos 2x = a$, show that $(1 + a)\tan^2 x - 4\tan x + a = 1$.

Hence, or otherwise, show that if $\tan x_1$ and $\tan x_2$ are the roots of the quadratic in part a, then $\tan(x_1 + x_2) = 2$.

21. a Solve $\left\{x^\circ : 3\sin x^\circ - \frac{1}{\sin x^\circ} = 2,\ 0 \le x \le 360\right\}$.

Hence, find $\left\{x^\circ : 3\sin x^\circ < \frac{1}{\sin x^\circ} + 2,\ 0 \le x \le 360\right\}$.

22. Prove that if $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \frac{\pi}{2}$, $\tan\alpha_1 < \frac{\sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n} < \tan\alpha_n$.

23. Using the inequality $\tan\frac{x}{2} > \frac{x}{2}$, prove that $\sin x > x - \frac{1}{4}x^3$, where $0 < x < \frac{\pi}{2}$.

24. Find all real values of x and y such that $\sin^4 x + \cos^4 y + 2 = 4\sin x\cos y$.

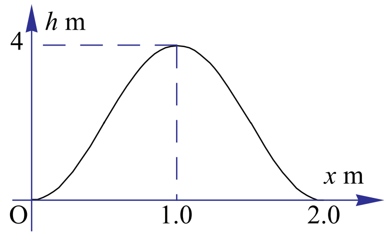
(Hint: let $u = \sin x$, $v = \cos y$ and show that $(u^2 - 1)^2 + (v^2 - 1)^2 + 2(u - v)^2 = 0$).

Exercise 3.6.2

14. A hill has its cross-section modelled by the function,

$$h : [0, 2] \mapsto \mathbb{R},\ h(x) = a + b\cos(kx),$$

where $h(x)$ measures the height of the hill relative to the horizontal distance x m from O.



a Determine the values of

i k

ii b

iii a

b How far, horizontally from O, would an ant climbing this hill from O be, when it first reaches a height of 1 metre?

c How much further, horizontally, will the ant have travelled when it reaches the same height of 1 metre once over the hill and on its way down?

15. A nursery has been infested by two insect pests: the Fruitfly and the Greatfly. These insects appear at about the same time that a particular plant starts to flower. The number of Fruitfly (in thousands), t weeks after flowering has started is modelled by the function

$$F(t) = 6 + 2\sin(\pi t),\ 0 \le t \le 4$$

Whereas the number of Greatfly (in thousands), t weeks after flowering has started is modelled by the function

$$G(t) = 0.25t^2 + 4,\ 0 \le t \le 4$$

a Copy and complete the following table of values, giving your answers correct to the nearest hundred.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$									
$G(t)$									

b On the same set of axes **draw** the graphs of:

i $F(t) = 6 + 2\sin(\pi t),\ 0 \le t \le 4$. ii $G(t) = 0.25t^2 + 4,\ 0 \le t \le 4$.

c On how many occasions will there be equal numbers of each insect?

d For what percentage of the time will there be more Greatflies than Fruitflies?

16. The depth, $d(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$d(t) = 12 + 3\sin\left(\frac{\pi}{6}t\right),\ 0 \le t \le 24$$

a Sketch the graph of $d(t)$ for $0 \le t \le 24$.

b For what values of t will: i $d(t) = 10.5$, $0 \le t \le 24$ ii $d(t) \ge 10.5$, $0 \le t \le 24$.

Boats requiring a minimum depth of b metres are only permitted to enter the harbour when the depth of water at the entrance of the harbour is at least b metres for a continuous period of one hour.

c Find the largest value of b, correct to two decimal place, which satisfies this condition.

Exercise 3.6.1

1 **a** $\frac{\pi}{4}, \frac{3\pi}{4}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{3}, \frac{2\pi}{3}$ **d** $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

e $\frac{\pi}{3}, \frac{5\pi}{3}$ **f** $\frac{5}{4}, \frac{7}{4}, \frac{13}{4}, \frac{15}{4}, \frac{21}{4}, \frac{23}{4}$

2 **a** $\frac{\pi}{4}, \frac{7\pi}{4}$ **b** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **c** $\frac{\pi}{6}, \frac{11\pi}{6}$ **d** π

e $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **f** $\frac{3}{2}, \frac{5}{2}, \frac{11}{2}$

3 **a** $\frac{\pi}{6}, \frac{7\pi}{6}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$ **d** $4\tan^{-1}2$

e $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$ **f** 3

4 **a** 90°, 330° **b** 180°,240° **c** 90°,270° **d** 65°,335°

e $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ **f** $0, \pi, 2\pi$ **g** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **h** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

5 **a** 60°,300° **b** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **c** $\frac{\pi}{6}, \frac{7\pi}{6}$ **d** 23°35',156°25'

e $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **g** $\frac{5\pi}{6}, \frac{9\pi}{6}$ **h** 3.3559^c, 5.2105^c

i $\frac{\pi}{3}, \frac{4\pi}{3}$ **j** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **k** $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$ **l** 68°12',248°12'

m $\frac{\pi}{3}, \frac{5\pi}{3}$ **n** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **o** ∅

6 **a** $-\frac{3\pi}{4}, \frac{\pi}{4}$ **b** $\pm\frac{\pi}{3}$ **c** $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$ **d** $-\frac{\pi}{2}$

e $\pm\frac{\pi}{2}$ **f** $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$ **g** $\frac{\pi}{2}, \frac{3\pi}{2}$ **h** $\frac{\pi}{2}, \frac{3\pi}{2}$

7 **a** $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{7\pi}{2}$ **b** $-2\pi,0,2\pi$

c $-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2}$ **d** $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$

e $2n\pi\pm\sin^{-1}\left(\frac{1}{3}\right)\pm\frac{\pi}{2},\frac{2(3n\pm1)\pi}{3},n=-1,3$

8 **a** $\frac{3\pi}{4},\frac{7\pi}{4},\tan^{-1}\left(\frac{2}{3}\right),\pi+\tan^{-1}\left(\frac{2}{3}\right)$ **b** $\frac{\pi}{3},\frac{2\pi}{3},\frac{3\pi}{4},\frac{4\pi}{3},\frac{5\pi}{3},\frac{7\pi}{4}$

9 **a** $\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12},\frac{19\pi}{12},\frac{23\pi}{12}$ **b** $\frac{2\pi}{3},\frac{4\pi}{3}$ **c** $0,1,2,3,4,5,6$

10 **a** $\frac{\pi}{3},\frac{5\pi}{3},\pi\pm\cos^{-1}\left(\frac{1}{4}\right)$ **b** $\frac{3\pi}{4},\frac{7\pi}{4},\tan^{-1}(3),\pi+\tan^{-1}(3)$ **c** $\frac{\pi}{6},\frac{7\pi}{6},\frac{\pi}{2},\frac{3\pi}{2}$

d $\tan^{-1}\left(\frac{3}{2}\right),\pi-\tan^{-1}(2),\pi+\tan^{-1}\left(\frac{3}{2}\right),2\pi-\tan^{-1}(2)$

11 **a** $2\sin\left(x+\frac{\pi}{6}\right)$ **b** $0,\frac{2\pi}{3},2\pi$

12 **a** $2\sin\left(x-\frac{\pi}{3}\right)$ **b** $\frac{\pi}{6},\frac{3\pi}{2}$

13 $\frac{\pi}{3},\frac{2\pi}{3}$

14 **a** $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)\cup\left(\frac{13\pi}{6},\frac{17\pi}{6}\right)$ **b** $\left(\pi+\sin^{-1}\left(\frac{1}{\sqrt{3}}\right),2\pi-\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)\cup\left(3\pi+\sin^{-1}\left(\frac{1}{\sqrt{3}}\right),4\pi-\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

15 **a ii** $\left[0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right]$ **b ii** $\left[0,\frac{\pi}{6}\right)\cup\left(\frac{\pi}{2},\frac{5\pi}{6}\right)\cup\left(\frac{3\pi}{2},2\pi\right]$

17 **a i** $\{x|x=k\pi+\alpha(-1)^k,k\in\mathbb{Z}\}$ **ii** $\{x|2k\pi+\alpha\le x\le(2k+1)\pi-\alpha,k\in\mathbb{Z}\}$

b $\left\{x|x=(2k+1)\frac{\pi}{5}\right\}\cup\{x|x=2k\pi\},k\in\mathbb{Z}$ **c** $\left\{x|x=\frac{2k\pi}{5}+\frac{\pi}{10}\right\}\cup\left\{x|x=2k\pi-\frac{\pi}{2}\right\},k\in\mathbb{Z}$

19 **a** $0,\frac{\pi}{3},\frac{5\pi}{3},2\pi$ **b** $\sqrt{2},\frac{\sqrt{2}}{2}$ **c** $2\cos\frac{\pi}{9},2\cos\frac{5\pi}{9},2\cos\frac{7\pi}{9}$

20 $\left\{\pm\frac{\pi}{4},\pm\frac{2\pi}{3},\pm\frac{3\pi}{4}\right\}$

22 **a** 90°,199°28',340°32' **b** (199°28',340°32')

25 $\left\{(x, y)|x = 2k\pi + \frac{\pi}{2}, y = 2k\pi\right\} \cup \left\{(x, y)|x = 2k\pi - \frac{\pi}{2}, y = 2k\pi + \pi\right\}, k \in \mathbb{Z}$

Exercise 3.6.2

1 **a** 5, 24, 11, 19 **b** $T = 5\sin\left(\frac{\pi t}{12} - 3\right) + 19$ **c** 23.6°

2 **a** 3, 4.2, 2, 7 **b** $L = 3\sin\left(\frac{\pi t}{2.1} - 3\right) + 7$

3 **a** 5, 11, 0, 7 **b** $V = 5\sin\left(\frac{2\pi t}{11}\right) + 7$

4 **a** 1, 11, 1, 12 **b** $P = \sin\frac{2\pi}{11}(t - 1) + 12$

5 **a** 2.6, 7, 2, 6 **b** $S = 2.6\sin\frac{2\pi}{7}(t - 2) + 6$

6 **a** 0.6, 3.5, 0, 11 **b** $P = 0.6\sin\left(\frac{4\pi t}{7}\right) + 11$

7 **a** 0.8, 4.6, 2.7, 11 **b** $D = 0.8\sin\frac{\pi}{2.3}(t - 2.7) + 11$

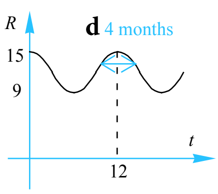
8 **a** 3000 **b** 1000, 5000 **c** $\frac{4}{9}$

9 **a** 6.5 m, 7.5 m **b** 1.58 sec, 3.42 sec

10 **a** 750, 1850 **b** 3.44 **c** mid-April to end of August

11 **a** 15000 **b** 12 months

c



d 4 months

12 **a** 2s **b** 26cm **c** 40s

d [18,34] **d** 8cm **e** 2s

f $D(t) = 8\sin\left(\pi\left(x + \frac{1}{2}\right)\right) + 26$ (for example) **g** 34cm

13 **a** $D(t)=20\sin\left(\frac{5\pi}{6}(x+0.2)\right)+52$ (for example) **b** 72cm

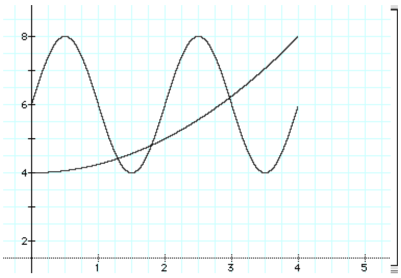
c 62cm **d** 0.86s

14 **a** π, −2, 2 **b** $\frac{1}{3}$m **c** $\frac{4}{3}$m

15 a

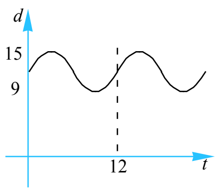
t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$	6	8	6	4	6	8	6	4	6
$G(t)$	4	4.0625	4.25	4.5625	5	5.5625	6.25	7.0625	8

b



c 3 **d** 38.4%

16 **a**

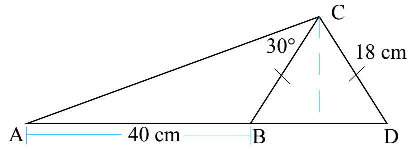


b i 7,11,19,23 **ii** $[0, 7] \cup [11, 19] \cup [23, 24]$ **c** 14.9 m

Exercise 3.7.3

8. The framework for an experimental design for a kite is shown. Material for the kite costs \$12 per square cm.

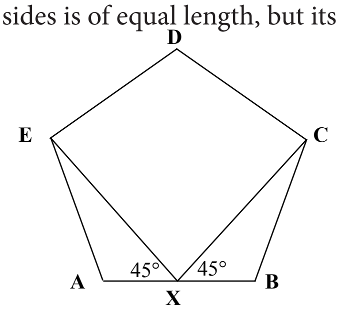
   How much will it cost for the material if it is to cover the framework of the kite.



9. A boy walking along a straight road notices the top of a tower at a bearing of 284°T. After walking a further 1.5 km he notices that the top of the tower is at a bearing of 293°T. How far from the road is the tower?

Exercise 3.7.7

117. A sandpit in the shape of a pentagon ABCDE is to be built in such a way that each of its sides is of equal length, but its angles are not all equal.



The pentagon is symmetrical about DX, where X is the midpoint of AB.

The angle AXE and BXC are both 45° and each side is 2 m long.

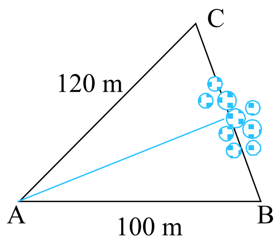
a Find $\angle XEA$.

b Find the length of EX.

c How much sand is required if the sandpit is 30 cm deep? Give your answer to three decimal places.

18. A triangular region has been set aside for a housing development which is to be divided into two sections. Two adjacent street frontages, AB and AC measure 100 m and 120 m respectively, with the 100 m frontage running in an easterly direction, while the 120 m frontage runs in a north-east direction. A plan for this development is shown alongside. Give all answers to the nearest metre.

Find the area covered by the housing development.



During the development stages, an environmental group specified that existing trees were not to be removed from the third frontage. This made it difficult for the surveyors to measure the length of the third frontage.

a Calculate the length of the third frontage, BC.

b The estate is to be divided into 2 regions, by bisecting the angle at A with a stepping wall running from A to the frontage BC.

c How long is this stepping wall?

Exercise 3.7.1

	a cm	b cm	c cm	A	B	C
1	13.3	37.1	48.2	10°	29°	141°
2	2.7	1.2	2.8	74°	25°	81°
3	11.0	0.7	11.3	60°	3°	117°
4	31.9	39.1	51.7	38°	49°	93°
5	18.5	11.4	19.5	68°	35°	77°
6	14.6	15.0	5.3	75°	84°	21°
7	26.0	7.3	26.4	79°	16°	85°
8	21.6	10.1	28.5	39°	17°	124°
9	0.8	0.2	0.8	82°	16°	82°
10	27.7	7.4	33.3	36°	9°	135°
11	16.4	20.7	14.5	52°	84°	44°
12	21.4	45.6	64.3	11°	24°	145°
13	30.9	27.7	22.6	75°	60°	45°
14	29.3	45.6	59.1	29°	49°	102°
15	9.7	9.8	7.9	65°	67°	48°
16	21.5	36.6	54.2	16°	28°	136°
17	14.8	29.3	27.2	30°	83°	67°
18	10.5	0.7	10.9	52°	3°	125°
19	11.2	6.9	17.0	25°	15°	140°
20	25.8	18.5	40.1	30°	21°	129

Exercise 3.7.2

	a	b	c	A°	B°	C°	c*	B*°	C*°
1	7.40	18.10	21.06	20.00	56.78	103.22	12.95	123.22	36.78
2	13.30	19.50	31.36	14.00	20.77	145.23	6.49	159.23	6.77
3	13.50	17.00	25.90	28.00	36.24	115.76	4.12	143.76	8.24
4	10.20	17.00	25.62	15.00	25.55	139.45	7.22	154.45	10.55
5	7.40	15.20	19.55	20.00	44.63	115.37	9.02	135.37	24.63
6	10.70	14.10	21.41	26.00	35.29	118.71	3.94	144.71	9.29
7	11.50	12.60	22.94	17.00	18.68	144.32	1.16	161.32	1.68
8	8.30	13.70	18.67	24.00	42.17	113.83	6.36	137.83	18.17
9	13.70	17.80	30.28	14.00	18.32	147.68	4.27	161.68	4.32
10	13.40	17.80	26.19	28.00	38.58	113.42	5.24	141.42	10.58
11	12.10	16.80	25.63	23.00	32.85	124.15	5.30	147.15	9.85
12	12.00	14.50	24.35	21.00	25.66	133.34	2.72	154.34	4.66
13	12.10	19.20	29.34	16.00	25.94	138.06	7.57	154.06	9.94
14	7.20	13.10	19.01	15.00	28.09	136.91	6.30	151.91	13.09
15	12.20	17.70	23.73	30.00	46.50	103.50	6.93	133.50	16.50
16	9.20	20.90	27.97	14.00	33.34	132.66	12.59	146.66	19.34
17	10.50	13.30	21.96	20.00	25.67	134.33	3.03	154.33	5.67
18	9.20	19.20	26.29	15.00	32.69	132.31	10.80	147.31	17.69
19	7.20	13.30	18.33	19.00	36.97	124.03	6.82	143.03	17.97
20	13.50	20.40	25.96	31.00	51.10	97.90	9.01	128.90	20.10

21 a–d no triangles exist.

Exercise 3.7.3

1 30.64 km

2 4.57 m

3 476.4 m

4 201°47'T

5 222.9 m **a** 3.40 m **b** 3.11 m

6 **b** 1.000 m **c** 1.715 m

7 **a** 51.19 min **b** 1 hr 15.96 min **c** 14.08 km

8 $4886

9 906 m

Exercise 3.7.4

a cm	b cm	c cm	A	B	C	
1	13.5	9.8	16.7	54°	36°	90°
2	8.9	10.8	15.2	35°	44°	101°
3	22.8	25.6	12.8	63°	87°	30°
4	21.1	4.4	21.0	85°	12°	83°
5	15.9	10.6	15.1	74°	40°	66°
6	8.8	13.6	20.3	20°	32°	128°
7	9.2	9.5	13.2	44°	46°	90°
8	23.4	62.5	58.4	22°	89°	69°
9	10.5	9.6	15.7	41°	37°	102°
10	21.7	36.0	36.2	35°	72°	73°
11	7.6	3.4	9.4	49°	20°	111°
12	7.2	15.2	14.3	28°	83°	69°
13	9.1	12.5	15.8	35°	52°	93°
14	14.9	11.2	16.2	63°	42°	75°
15	2.0	0.7	2.5	38°	13°	129°
16	7.6	3.7	9.0	56°	24°	100°
17	18.5	9.8	24.1	45°	22°	113°
18	20.7	16.3	13.6	87°	52°	41°
19	14.6	22.4	29.9	28°	46°	106°
20	7.0	6.6	9.9	45°	42°	93°
21	21.8	20.8	23.8	58°	54°	68°
22	1.1	1.7	1.3	41°	89°	50°
23	1.2	1.2	0.4	85°	76°	19°
24	23.7	27.2	29.7	49°	60°	71°
25	3.4	4.6	5.2	40°	60°	80°

Exercise 3.7.5

1 **a** 10.14 km **b** 121°T

2 7° 33'

3 4.12 cm

4 57.32 m

5 315.5 m

6 **a** 124.3 km **b** W28° 47’ S

Exercise 3.7.6

1

a	1999.2 cm^2	**b**	756.8 cm^2	**c**	3854.8 cm^2	**d**	2704.9 cm^2
e	538.0 cm^2	**f**	417.5 cm^2	**g**	549.4 cm^2	**h**	14.2 cm^2
i	516.2 cm^2	**j**	281.5 cm^2	**k**	918.8 cm^2	**l**	387.2 cm^2
m	139.0 cm^2	**n**	853.7 cm^2	**o**	314.6 cm^2		

2 69 345 m^2

3 $100\pi - 6\sqrt{91}$ cm^2

4 17.34 cm

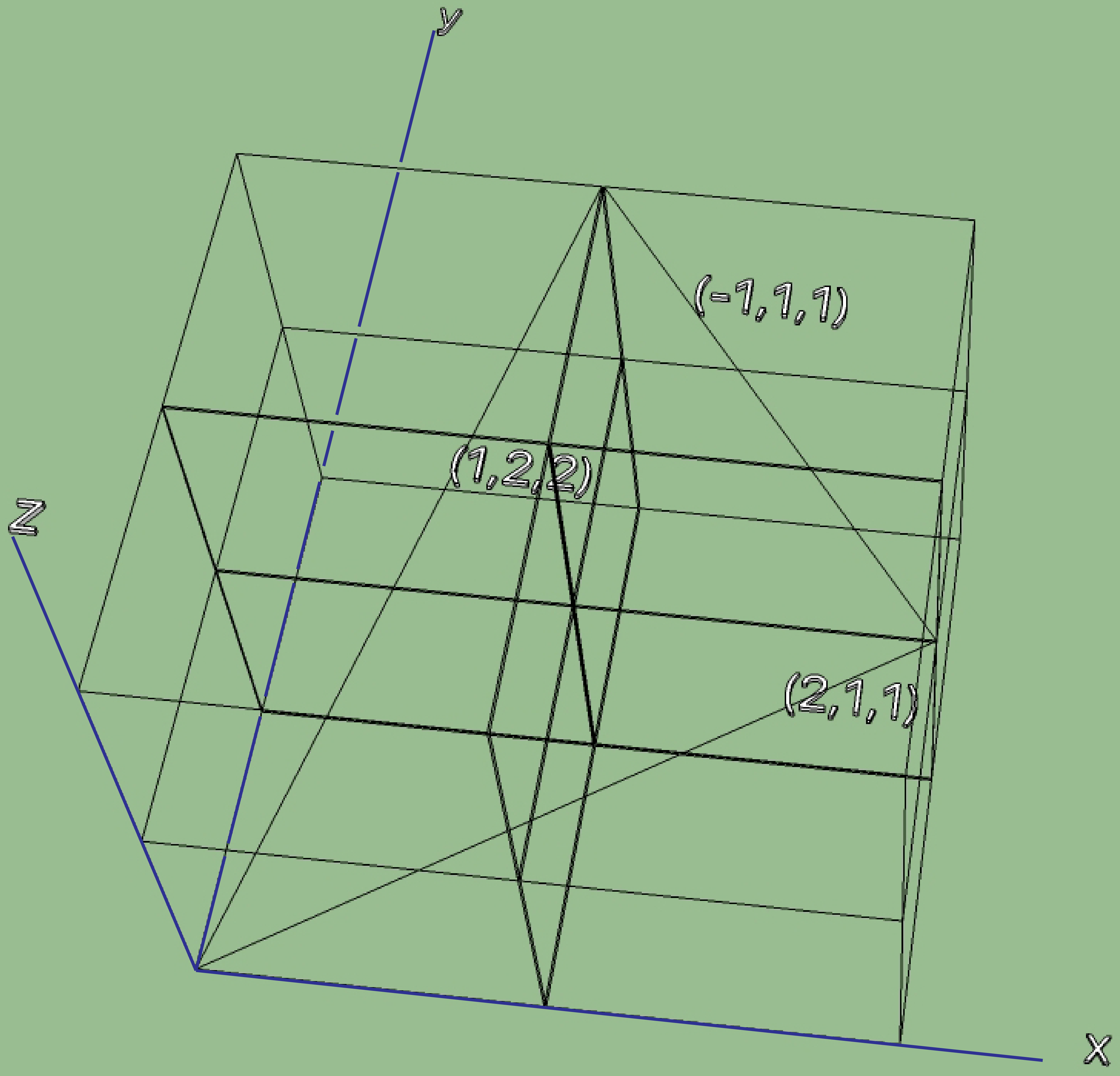
5 **a** 36.77sq units **b** 14.70 sq units **c** 62.53 sq units

6 52.16 cm^2

7 7° 2’

8 $\frac{(b + a \times \tan\theta)^2}{2\tan\theta}$

9 Area of $\Delta ACD = 101.78$ cm^2 , Area of $\Delta ABC = 61.38$ cm^2


y
(-1,1,1)
(1,2,2)
z
(2,1,1)
x

Exercise 4.1.1

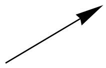
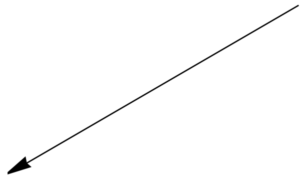
11. Patrick walks for 200 m to point P due east of his cabin at point O, then 300 m due north where he reaches a vertical cliff, point Q. Patrick then climbs the 80 m cliff to point R.

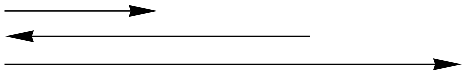
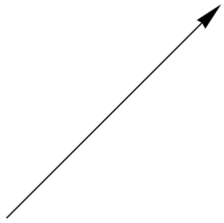
a Draw a vector diagram showing the vectors **OP**, **PQ** and **QR**.

b Find: i |**OQ**| ii |**OR**|

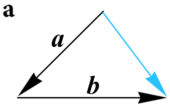
Exercise 4.1.1

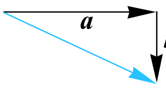
1 **a** **b** 

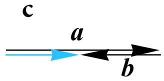
c  **d** 

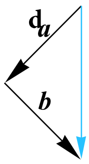
2 **a** **b** **c**  **d** 

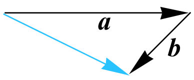
3	**a**	{a,b,e,g,u}; {d,f}	**b**	{d,f}; {a,c}; {b,e}	**c**	{a,g},{c,g}
	d	{d,f}, {b,e}	**e**	{d,f}, {b,e}, {a,c,g}		

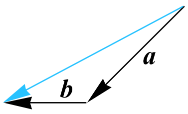
4 a 

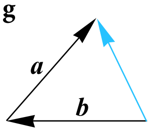
b 

c 

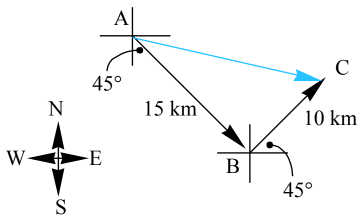
d 

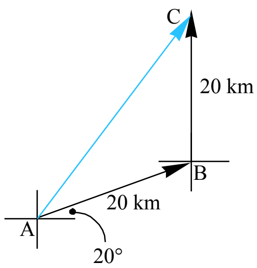
e 

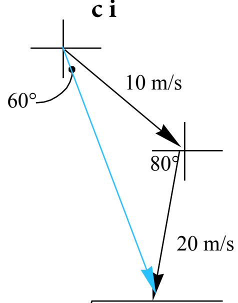
f 

g 

5	**a**	AC	**b**	AB	**c**	AD	**d**	BA	**e**	**0**
6	**a**	Y	**b**	N	**c**	Y	**d**	Y	**e**	N

7 a i 

b i 

c i 

ii $\sqrt{325}$ **iv** $20\sqrt{2(1-\cos 110°)}$ **v** $10\sqrt{5-4\cos 110°}$

8 72.11 N, E $33°41'$ N

9 2719 N along river

10 **b i** 200 kph N **ii** 213.6 kph, N $7°37'$ W

11 **b i** 200 **ii** 369.32

Exercise 4.1.2

1 **a** $4i + 28j - 4k$ **b** $12i + 21j + 15k$ **c** $-2i + 7j - 7k$ **d** $-6i - 12k$

2 **a** $3i - 4j + 2k$ **b** $-8i + 24j + 13k$ **c** $18i - 32j + k$ **d** $-15i + 36j + 12k$

3 **a** $\begin{pmatrix} 11 \\ 0 \\ 8 \end{pmatrix}$ **b** $\begin{pmatrix} -27 \\ 1 \\ -22 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ -6 \\ 12 \end{pmatrix}$ **d** $\begin{pmatrix} 16 \\ -1 \\ 14 \end{pmatrix}$

4 $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$

5 $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $(-2, 3)$

6 **a** $8i - 4j - 28k$ **b** $-19i - 7j - 16k$ **c** $-17i + j + 22k$ **d** $40i + 4j - 20k$

7 **a** $\begin{pmatrix} 20 \\ 1 \\ 25 \end{pmatrix}$ **b** $\begin{pmatrix} 12 \\ 2 \\ 16 \end{pmatrix}$ **c** $\begin{pmatrix} -4 \\ -38 \\ -32 \end{pmatrix}$ **d** $\begin{pmatrix} -20 \\ -22 \\ -40 \end{pmatrix}$

8 $A = -4$, $B = -7$

9 **a** $(2, -5)$ **b** $(-4, 3)$ **c** $(-6, -5)$

10 Depends on basis used. Here we used: East as $\boldsymbol{i}$, North $\boldsymbol{j}$ and vertically up $\boldsymbol{k}$

b $D = 600i - 800j + 60k$, $A = -1200i - 300j + 60k$ **c** $1800i - 500j$

Exercise 4.2.1

1	**a**	4	**b**	–11.49	**c**	25		
2	**a**	12	**b**	27	**c**	–8	**d**	–49
	f	4	**g**	–21	**h**	6	**i**	–4
3	**a**	79°	**b**	108°	**c**	55°	**d**	50°
	e	74°	**f**	172°	**g**	80°	**h**	58°
4	**a**	–8	**b**	0.5				
5	**a**	–6	**b**	2	**c**	Not possible	**d**	5
	e	Not possible	**f**	0				
6	**a**	$4-2\sqrt{3}$	**b**	$2\sqrt{3}-4$	**c**	$14-2\sqrt{3}$	**d**	Not possible

7 1

8 105.2°

9 $x=-\frac{16}{7}, y = -\frac{44}{7}$

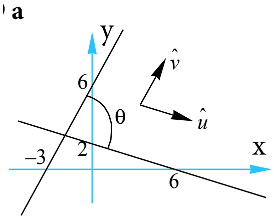
10 $\pm\frac{1}{\sqrt{11}}(-\boldsymbol{i}+\mathbf{j}+3\boldsymbol{k})$

12 **a** $\lambda(-16\boldsymbol{i}-10\boldsymbol{j}+\boldsymbol{k})$ **b** e.g. $\boldsymbol{i}+\boldsymbol{j}+\frac{3}{7}\boldsymbol{k}$

14 $\boldsymbol{a}\perp\boldsymbol{b}-\boldsymbol{c}$ if $\boldsymbol{b}\neq\boldsymbol{c}$ or $\boldsymbol{b} = \boldsymbol{c}$

16 **a** $\frac{1}{3}$ **b** $\frac{1}{\sqrt{3}}$

17 **a**



b i $\hat{\boldsymbol{u}} = \frac{1}{\sqrt{10}}(3\boldsymbol{i}-\mathbf{j})$ **ii** $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{5}}(\boldsymbol{i}+2\mathbf{j})$

c 81.87°

18 $\frac{1}{2}(-\boldsymbol{i}+\mathbf{j}+\sqrt{2}\boldsymbol{k})$

23 **a** Use $\boldsymbol{i}$ as a 1 km eastward vector and $\boldsymbol{j}$ as a 1 km northward vector.

b $\overrightarrow{WD} = 4\boldsymbol{i}+8\boldsymbol{j}$, $\overrightarrow{WS} = 13\boldsymbol{i}+\boldsymbol{j}$ and $\overrightarrow{DS} = 9\boldsymbol{i}-7\boldsymbol{j}$ **c** $\frac{1}{\sqrt{80}}(4\boldsymbol{i}+8\boldsymbol{j})$

d $\frac{d}{\sqrt{80}}(4\boldsymbol{i}+8\boldsymbol{j})$ **e** $3\boldsymbol{i}+6\boldsymbol{j}$

Example

Find the angle between the lines:

$$r_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}, r_2 = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

It is necessary to find the angle between the two vectors that represent the directions of the lines:

These are: $\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Using the 'dot product method':

$$\left| \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\left| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 \times 1 + 3 \times 2 = 5$$

$$\cos\theta = \frac{5}{\sqrt{10} \times \sqrt{5}}$$

$$\theta = 45^\circ$$

Exercise 4.3.1

11. The line L is defined by the parametric equations $x = 4 - 5k$ and $y = -2 + 3k$.

a Find the coordinates of three points on L.

b Find the value of k that corresponds to the point (14, –8).

c Show that the point (–1, 4) does not lie on the line L.

d Find the vector form of the line L.

e A second line, M, is defined parametrically by $x = a + 10\lambda$ and $y = b - 6\lambda$. Describe the relationship between M and L for the case that:

i $a = 8$ and $b = 4$ ii $a = 4$ and $b = -2$

12. Find the Cartesian equation of the line that passes through the point A(2, 1) and such that it is perpendicular to the vector $4\boldsymbol{i} + 3\boldsymbol{j}$.

13. Find the direction cosines for each of the following lines:

a $\boldsymbol{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ **b** $\boldsymbol{r} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

14. a Show that the line $ax + by + c = 0$ has a directional vector $\begin{pmatrix} b \\ -a \end{pmatrix}$ and a normal vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

b By making use of directional vectors, which of the following lines are parallel to $L : 2x + 3y = 10$?

i $5x - 2y = 10$

ii $6x + 9y = 20$

iii $4x + 6y = -10$

15. Find the point of intersection of the lines $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ and $\frac{x-3}{2} = \frac{y}{5}$.

16. Find a vector equation of the line passing through the origin that also passes through the point of intersection of the lines:

$$\boldsymbol{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

17. Consider the line with vector equation $\boldsymbol{r} = (4\boldsymbol{i} - 3\boldsymbol{j}) + \lambda(3\boldsymbol{i} + 4\boldsymbol{j})$. Find the points of intersection of this line with the line:

a $\boldsymbol{u} = (4\boldsymbol{i} + 5\boldsymbol{j}) + \mu(2\boldsymbol{i} - \boldsymbol{j})$

b $\boldsymbol{v} = (-2\boldsymbol{i} + 3\boldsymbol{j}) + t(-6\boldsymbol{i} - 8\boldsymbol{j})$

c $\boldsymbol{w} = (13\boldsymbol{i} + 9\boldsymbol{j}) + s(3\boldsymbol{i} + 4\boldsymbol{j})$

Exercise 4.3.2

8. Show that the lines $\frac{x-1}{2}=2-y=5-z$ and $\frac{4-x}{4}=\frac{3+y}{2}=\frac{5+z}{2}$ are parallel.

9. Find the Cartesian equation of the lines joining the points

a (–1, 3, 5) to (1, 4, 4) b (2, 1, 1) to (4, 1, –1)

10. a Find the coordinates of the point where the line $\boldsymbol{r} = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ intersects the *x*-*y* plane.

b The line $\frac{x-3}{4}=y+2=\frac{4-z}{5}$ passes through the point (*a*, 1, *b*). Find the values of *a* and *b*.

11. Find the Cartesian equation of the line having the vector form:

a $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ b $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

In each case, provide a diagram showing the lines.

12. Find the vector equation of the line represented by the Cartesian form $\frac{x-1}{2} = \frac{1-2y}{3} = z-2$.

Clearly describe this line.

13. Find the acute angle between the following lines.

a $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

b $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c $\frac{x-3}{-1} = \frac{2-y}{3} = \frac{z-4}{2}$ and $\frac{x-1}{2} = \frac{y-2}{-2} = z-2$

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14. Find the point of intersection of the lines:

a $\frac{x-5}{-2} = y-10 = \frac{z-9}{12}$ and $x = 4, \frac{y-9}{-2} = \frac{z+9}{6}$

b $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$ and $\frac{2-x}{4} = \frac{y+3}{2} = \frac{4-2z}{1}$

15. Find the Cartesian form of the lines with parametric equation given by:

L : $x = \lambda, y = 2\lambda+2, z = 5\lambda$ and M : $x = 2\mu-1, y = -1+3\mu, z = 1-2\mu$

a Find the point of intersection of these two lines.

b Find the acute angle between these two lines.

c Find the coordinates of the point where: i L cuts the x-y plane. ii M cuts the y-z plane.

16. Show that the lines $\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z+1}{5}$ and $\frac{x-5}{-3} = \frac{y-1}{2} = \frac{z-4}{-5}$ are coincident.

17. Show that the lines $\frac{x-1}{-3} = y-2 = \frac{7-z}{11}$ and $\frac{x-2}{3} = \frac{y+1}{8} = \frac{z-4}{-7}$ are skew.

18. Find the equation of the line passing through the origin and the point of intersection of the lines with equations $x-2 = \frac{y-1}{4}, z = 3$ and $\frac{x-6}{2} = y-10 = z-4$.

19. The lines $\frac{x}{3} = \frac{y-2}{4} = 3+z$ and $x = y = \frac{z-1}{2k}$, $k \in \mathbb{R}\backslash\{0\}$ meet at right angles. Find k.

20. Consider the lines L : $x = 0, \frac{y-3}{2} = z+1$ and M : $\frac{x}{4} = \frac{y}{3} = \frac{z-10}{-1}$.

Find, correct to the nearest degree, the angle between the lines L and M.

21. Find the value(s) of k, such that the lines $\frac{x-2}{k} = \frac{y}{2} = \frac{3-z}{3}$ and $\frac{x}{k-1} = \frac{y+2}{3} = \frac{z}{4}$ are perpendicular.

22. Find a direction vector of the line that is perpendicular to both $\frac{x+1}{3} = \frac{y+1}{8} = \frac{z+1}{12}$ and $\frac{1-2x}{-4} = \frac{3y+1}{9} = \frac{z}{6}$.

23. Are the lines $\frac{x-1}{5} = \frac{y+2}{4} = \frac{4-z}{3}$ and $\frac{x+2}{3} = \frac{y+7}{2} = \frac{2-z}{3}$ parallel? Find the point of intersection of these lines.

What do you conclude?

Exercise 4.3.1

1 **a i** $r = i + 2j$ **ii** $r = -5i + 11j$ **iii** $r = 5i - 4j$ **b** line joins (1, 2) and (5, –4)

2 **a** $r = 2i + 5j + \lambda(3i - 4j)$ **b** $r = -3i + 4j + \lambda(-i + 5j)$ **c** $r = j + \lambda(7i + 8j)$

d $r = i - 6j + \lambda(2i + 3j)$ **e** $r = \begin{pmatrix}-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\10\end{pmatrix}$ or $r = -i - j + \lambda(-2i + 10j)$

f $r = \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}$ or $r = i + 2j + \lambda(5i + j)$

3 **a** $r = 2i + 3j + \lambda(2i + 5j)$ **b** $r = i + 5j + \lambda(-3i - 4j)$ **c** $r = 4i - 3j + \lambda(-5i + j)$

4 **a** $r = 9i + 5j + \lambda(i - 3j)$ **b** $r = 6i - 6j + t(-4i - 2j)$

c $r = -i + 3j + \lambda(-4i + 8j)$ **d** $r = i + 2j + \mu\left(\frac{1}{2}i - \frac{1}{3}j\right)$

5 **a** $x = -8 + 2\mu$, $y = 10 + \mu$ **b** $x = 7 - 3\mu$, $y = 4 - 2\mu$

c $x = 5 + 2.5\mu$, $y = 3 + 0.5\mu$ **d** $x = 0.5 - 0.1t$, $y = 0.4 + 0.2t$

6 **a** $\frac{x-1}{3} = y - 3$ **b** $\frac{x-2}{-7} = \frac{y-4}{-5}$

c $x + 2 = \frac{y+4}{8}$ **d** $x - 0.5 = \frac{y-0.2}{-11}$

e $x = 7$

7 **a** $r = 2j + t(3i + j)$ **b** $r = 5i + t(i + j)$ **c** $r = -6i + t(2i + j)$

8 **a** $6i + 13j$ **b** $-\frac{16}{3}i - \frac{28}{3}j$

9 $r = 2i + 7j + t(4i + 3j)$

11 **a** (4, –2), (–1, 1), (9, –5) **b** –2 **d** $r = 4i - 2j + \lambda(-5i + 3j)$ **e i** $M \parallel L$ **ii** $M = L$

12 $4x + 3y = 11$

13 **a** $\frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}$ **b** $\frac{4}{5}, \frac{3}{5}$

14 **b** ii and iii

15 (–83, –215)

16 $r = \frac{k}{7}(19\boldsymbol{i} + 20\boldsymbol{j})$

17 **a** $\left(\frac{92}{11}, \frac{31}{11}\right)$ **b** Ø **c** Lines are coincident, all points are common.

Exercise 4.3.2

1 **a** $r = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$ **b** $r = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{k})$

2 **a** $r = 2\boldsymbol{i} + 5\boldsymbol{k} + t(\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$ **b** $r = 3\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k} + t(4\boldsymbol{i} + 9\boldsymbol{j} - 5\boldsymbol{k})$

c $r = 4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k} + t(7\boldsymbol{i} + 7\boldsymbol{k})$

3 **a** $\frac{x}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ **b** $\frac{x+2}{5} = \frac{z+1}{-2}, y = 3$ **c** $x = y = z$

4 $x = 5 - 7t$, $y = 2 + 2t$, $z = 6 - 4t$ $\quad \boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} + t\begin{pmatrix} -7 \\ 2 \\ -4 \end{pmatrix}$ $\quad \frac{x-5}{-7} = \frac{y-2}{2} = \frac{z-6}{-4}$

5 $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

6 **a** $x = 2 + 3t$, $y = 5 + t$, $z = 4 + 0.5t$ **b** $x = 1 + 1.5t$, $y = t$, $z = 4 - 2t$

c $x = 3 - t$, $y = 2 - 3t$, $z = 4 + 2t$ **d** $x = 1 + 2t$, $y = 3 + 2t$, $z = 2 + 0.5t$

7 **a** $\frac{x-4}{3} = \frac{y-1}{-4} = \frac{z+2}{-2}$ **b** $x = 2, y = \frac{z-1}{-3}$

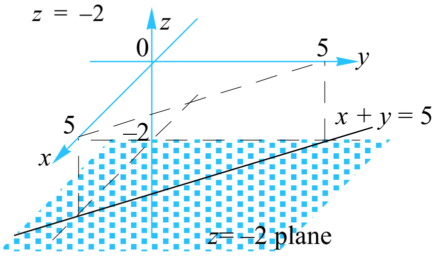
9 **a** $\frac{x+1}{2} = y - 3 = \frac{z-5}{-1}$ **b** $\frac{x-2}{2} = \frac{z-1}{-2}, y = 1$

10 **a** $(1, -1, 0)$ **b** $a = 15, b = -11$

11 a $x = 1 + t$

$y = 4 - t$

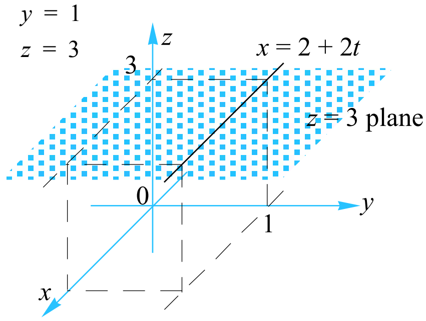
$z = -2$



b $x = 2 + 2t$

$y = 1$

$z = 3$



12 $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix}$. Line passes through (1, 0.5, 2) and is parallel to the vector $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

13 **a** 54.74° **b** 82.25° **c** 57.69°

14 **a** (4, 10.5, 15) **b** Does not intersect.

15 **a** L: $x = \frac{y-2}{2} = \frac{z}{5}$, M: $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{-2}$ **b** Ø **c** 84.92°

d i (0, 2, 0) **ii** $\left(0, \frac{1}{2}, 0\right)$

18 $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$

19 $k = -\frac{7}{2}$

20 64°

21 3 or –2

22 $12\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$ (or any multiple thereof)

23 Not parallel. Do not intersect. Lines are skew.

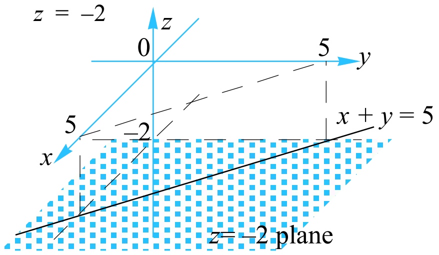
Exercise 4.4.1

1 **a** $\frac{x+1}{2} = y-3 = \frac{z-5}{-1}$ **b** $\frac{x-2}{2} = \frac{z-1}{-2}, y = 1$

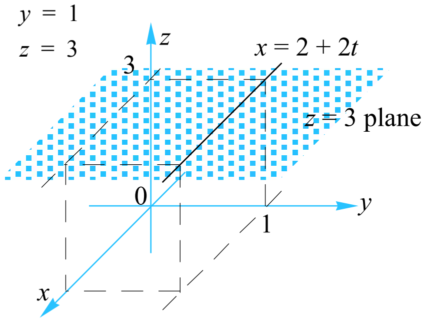
2 **a** $(1, -1, 0)$ **b** $a = 15, b = -11$

3

a $x = 1+t$
$y = 4-t$
$z = -2$



b $x = 2+2t$
$y = 1$
$z = 3$



4 $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix}$. Line passes through (1, 0.5, 2) and is parallel to the vector $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

5 **a** 54.74° **b** 82.25° **c** 57.69°

6 **a** (4, 10.5, 15) **b** Does not intersect.

7 **a** $L: x = \frac{y-2}{2} = \frac{z}{5}, M: \frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{-2}$ **b** $\varnothing$ **c** 84.92°

d i $(0, 2, 0)$ **ii** $\left(0, \frac{1}{2}, 0\right)$

10 $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$

11 $k = -\frac{7}{2}$

12 64°

13 3 or −2

14 $12\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$ (or any multiple thereof)

15 Not parallel. Do not intersect. Lines are skew.

16 $t=2$

17 (1,1,1), no

18 ~21

Exercise 4.5.2

10. Prove that $\boldsymbol{a} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = 0$.

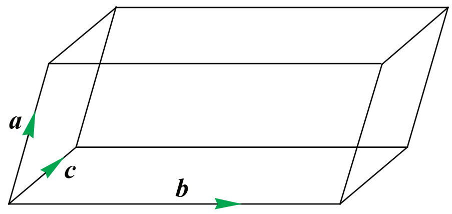
11. Prove that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$.

12. What condition must the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy in order that the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$ are collinear?

13. Prove that if $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{0}$ then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Exercise 4.5.3

12. Prove that the volume of the parallelepiped determined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is given by $|\boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c})|$.



Find the volume of the parallelepiped determined by the vectors:

$\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = -2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$.

13.

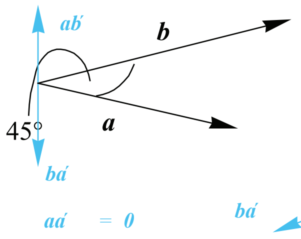
a Consider the triangle ABC where the points M, N and P lie on the sides [AB], [BC] and [CA] respectively and are such that $\mathbf{AM} = k_1\mathbf{AB}$, $\mathbf{BN} = k_2\mathbf{BC}$ and $\mathbf{CP} = k_3\mathbf{CA}$, where $k_1, k_2, k_3 \in \mathbb{R}$. Show that if the vectors $\mathbf{CM}$, $\mathbf{AN}$ and $\mathbf{BP}$ form a triangle, then, $k_1 = k_2 = k_3$.

b Consider the triangle ABC where the points M, N and P lie on the sides [AB], [BC] and [CA] respectively and are such that $\mathbf{AM} = k\mathbf{AB}$, $\mathbf{BN} = k\mathbf{BC}$ and $\mathbf{CP} = k\mathbf{CA}$, and $k \in \mathbb{R}$. Find the value of k so that the area of the triangle formed by the vectors $\mathbf{CM}$, $\mathbf{AN}$ and $\mathbf{BP}$ is a minimum.

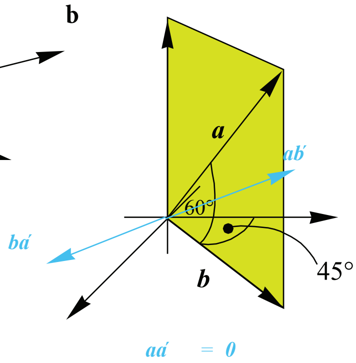
Exercise 4.5.1

1 **a** 5 **b** $4\sqrt{3}$ **c** 0 **d** 6 **e** 0

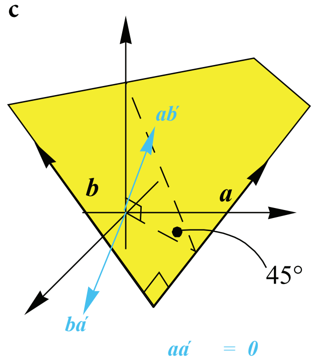
2 **a**



b



c



3 **a** $2\sqrt{91}$ **b** 16

4 $56°27'$

5 $3\sqrt{5}$

6 **a** $2\sqrt{3}$ **b** $5\sqrt{3}$

Exercise 4.5.2

1 **a** $-12\boldsymbol{i}+4\boldsymbol{k}$ **b** $10\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k}$ **c** $18\boldsymbol{i}-9\boldsymbol{j}$ **d** $10\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$

e $-6\boldsymbol{i}+9\boldsymbol{j}+8\boldsymbol{k}$ **f** $20\boldsymbol{i}-13\boldsymbol{j}-4\boldsymbol{k}$

2 $-10\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$

5 **a i** 0 **ii** 0

6 $\dfrac{-6\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{11}}$

7 **a** $\lambda\boldsymbol{k}$ **b** $\lambda(9\boldsymbol{i}-3\boldsymbol{j}+9\boldsymbol{k})$

8 **a** 90° **b** 79.1°

12 They must be parallel.

Exercise 4.5.3

1 **a** $\sqrt{54}$ **b** $\sqrt{234}$

2 **a** $\frac{1}{2}(-3\boldsymbol{i}-13\boldsymbol{j}+29\boldsymbol{k})$ **b** $\frac{1}{2}\sqrt{1019}$ **c** 67.84°

3 $\frac{1}{2}\sqrt{2331}$

4 $12\sqrt{2}$

5 $43°36'$

6 $\sqrt{293}$ sq. units

7 $\frac{1}{2}\sqrt{35}$

9 **a** $\mathbf{OA} = \cos\alpha \boldsymbol{i} + \sin\alpha \boldsymbol{j}$, $\mathbf{OB} = \cos\beta \boldsymbol{i} + \sin\beta \boldsymbol{j}$

12 66 cubic units

13 **b** $k = 0.5$

Exercise 4.6.1

1 **a** $r = i + k + \lambda(3i + 2j + k) + \mu(-2i - j + k)$ **b** $r = -i + 2j + k + \lambda(i - j + 2k) + \mu(-i - j + k)$

c $r = 4i + j + 5k + \lambda(2i + 2j - k) + \mu(2i - j + 3k)$ **d** $r = 2i - 3j - k + \lambda(-3i + j - 2k) + \mu\left(i - 2j + \frac{1}{2}k\right)$

2 **a** $3x - 5y + z = 4$ **b** $x - 3y - 2z = -9$ **c** $5x - 8y - 6z = -18$

d $7x + y - 10z = 21$

3 **a i** $\mathbf{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}-3\\-1\\-3\end{pmatrix} + \mu\begin{pmatrix}-2\\2\\2\end{pmatrix}$ **ii** $x + 3y - 2z = 3$

b i $\mathbf{r} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-2\\5\\-11\end{pmatrix} + \mu\begin{pmatrix}-1\\4\\-1\end{pmatrix}$ **ii** $13x + 3y - z = 31$

4 **a i** $\mathbf{r} = \begin{pmatrix}2\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\1\\2\end{pmatrix}$ **ii** $\mathbf{r} = \lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\1\\2\end{pmatrix}$

b i $x + 7y - 5z = -27$ **ii** $x + 7y - 5z = 0$ **c** $-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$

d Coefficients are the negative of those in part **b.**

Exercise 4.6.2

1 **a** $2x - y + 5z = 7$ **b** $-4x + 6y - 8z = 34$ **c** $-x + 3y - 2z = 0$

d $5x + 2y + z = 0$

2 c and d

3 **a** $-3x - y + 2z = 3$ **b** $y = 2$ **c** $2x + 2y - z = -3$

4 **a** 29.5° **b** 70° **c** 90° **d** 11°

5 **a** 83° **b** 50° **c** 49°

6 **a** $2x + y + 2z = 12$ **b** $8x + 17y - z = 65$

7 $x - 2y + 3z = -2$

8 $3x - 2y + 5z = -2$

9 $a = \frac{24}{13}, b = \frac{18}{13}$

10 **a** $r = 3i + 2j + k + t(2i + 5j + 5k)$ **b** $3i + 2j + k$ **c** 49.8°

Exercise 4.6.3

1 **a** $2x + 7y - 6z = 33$; $\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ 7 \\ -6 \end{pmatrix} = 33$ **b** $x - 2y = 0$; $\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 0$

2 $3x - y - z = 2$

5 **a** 3 **b** $\frac{5}{3}$ **c** $\sqrt{11}$ **d** $\frac{20}{\sqrt{21}}$

6 $3x + 4y - 5z = -4$

7 $2x + 3y - 3z = 5$

9 $x + 5y - 6z = -19$

Exercise 4.7.1

1 **a** (7, 5, –3)

2 Lines that intersect are **b** and **c**; (7, –4, 10); 46.7°

3 (5, –2, –3)

4 (4, 0, 6)

Exercise 4.7.2

1 **a i** (7, 4, 2) **ii** 36.3° **b i** (5, 2, –5) **ii** 10.1°

c i (6, –5, –7) **ii** 4.4° **d i** (3, –1, 1) **ii** 29.1°

2 **a** $\left(\frac{3}{2}, \frac{5}{2}, 2\right)$ **b** (0, 4, 1)

3 **a** Plane is parallel to the z-axis slicing the x-y plane on the line $x + y = 6$.

b $x = 4$ forms a plane. $y = 2z$ is in this plane parallel to the y-z plane. (4, 2, 1)

4 13

Exercise 4.7.3

1 **a** $x = -2y + 9 = -2z - 3$ or $\frac{x+3}{-2} = y - 6 = z$; 22°12′

b $\frac{21 - 10x}{-9} = y = \frac{7 - 10z}{7}$ or $\frac{7x - 29}{-11} = \frac{7y + 9}{-6} = z$; 70°48′ **c** planes parallel

d $x = 4, z = -2 - 2y$; 65°54′

3 73°42′

Exercise 4.7.4

1 e.g. the faces of a triangular prism.

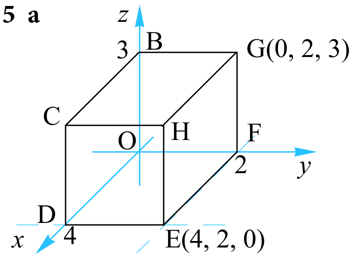
2 **a** $4 - 2x = \frac{2y - 4}{-5} = z$ or $\frac{x}{1} = \frac{y + 8}{5} = \frac{z - 4}{-2}$

3 **a** $5 - 4x = y = \frac{8 - 4z}{7}$ or $\frac{x}{1} = \frac{y - 5}{-4} = \frac{z + 6}{7}$

4 **a** No solution **b** Unique solution (5, 1, 4) **c** Unique solution (5, 1, –3)

d Intersect on plane $\frac{5x+19}{-8} = \frac{5y-13}{1} = z$

5 a



b $x = 4t \quad x = 4s$; $y = 2t \quad y = 2s$; $z = 3t \quad z = 3 - 3s$ (2, 1, 1.5)

c $3x + 6y - 4z = 12$

d $\left(\frac{8}{3}, \frac{4}{3}, 1\right)$ 58°52′ **e** 59.2°

6 None of these planes is parallel but the lines of intersection of pairs of planes are skew.

7 $k = 2$; $\boldsymbol{r} = \begin{pmatrix}0\\3.5\\1.5\end{pmatrix} + t\begin{pmatrix}1\\-2.5\\-0.5\end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix}3\\-4\\0\end{pmatrix} + \lambda\begin{pmatrix}-2\\5\\1\end{pmatrix}$

8 **a** $-2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ **b** $\boldsymbol{r} = t\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ **c i** 5 **iii** not 5

9 **b** $(a - b + c, a + b - c, -a + b + c)$ **c** $\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}, \frac{1}{a} + \frac{1}{b} - \frac{1}{c}, -\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

10 **a** 2, 3 **b** 3 **c** For $k = 2$, $x - 1 = 4 - y = z$

Exercise 5.1.3

6. For the figures given below, calculate the mean from the original data.

5	16	15	17	9	16	19	15
6	17	10	16	8	13	13	19
7	16	18	18	8	18	19	18
6	17	19	16	7	13	17	19
9	14	17	19	9	16	17	19
8	18	16	15	8	18	16	15

a Use the frequency table method with class intervals 4–6, 7–9 etc. to calculate the mean of the data.

b Use the frequency table method with class intervals 1–5, 6–10 etc. to calculate the mean of the data.

7. The failure times for electronic components, labelled A and B, are considered by a manufacturer of computers. The supplier of these components carries out tests on a sample of each type, resulting in the following observations:

Failure times

Time to failure (hours)	Type A	Type B
[0, 10[	15	6
[10, 20[	15	7
[20, 30[	19	13
[30, 40[	19	19
[40, 50[	17	33
[50, 60[	18	28
[60, 70[	16	21
[70, 80[	18	23
[80, 90[	15	18
[90, 100[	13	15

a Draw a histogram for each of the data sets.

b Determine the mean times to failure for each type of component.

c Which of Type A and Type B would you recommend the computer manufacturer purchase?

8. Weekly sale figures for phone cards at a local store are shown below.

Phone card sale figures

Number of cards	Number
0–4	10
5–9	13
10–14	9
15–19	14
20–24	8

Calculate the mean number of cards that are sold each week at this store.

9. The data set A has a mean of 16 while that of set B has a mean of 20. Calculate the values of a and b.

Set A:	15	11	24	18	19	15	14	19	a	$3b$
Set B:	$2a$	15	25	20	17	18	22	24	b	24

10. Tax refunds to the 200 workers from a small town have been allocated according to the following table.

Tax refunds

Refund	Frequency
[3000, 4000[	20
[4000, 5000[	36
[5000, 6000[	62
[6000, 7000[	32
[7000, 8000[	24
[8000, 9000[	12
[9000, 10,000[	8
[10,000, 11,000[	6

For the table shown draw its: **i** histogram **ii** cumulative frequency graph.

Calculate the mean tax refunds for this town.

Exercise 5.1.4

9. Two sets of records have the following score: A: $x + 1, x + 4, x + 6, x - 5, x + 4$

B: $y + 2, y - 2, y, y + 2, y + 3$

The mean score for set A is found to be 22. What is the value of x?

If $\bar{x} > \bar{y}$, what is the largest integer value that y can be?

10. A set of digits consists of x – 0s and y – 1s.

Calculate the mean for this data set and show that the standard deviation is $\dfrac{\sqrt{xy}}{x+y}$.

Exercise 5.1.1

1	a i	14 500	ii	2 000	b	305 (304.5)		

2 Sample size is large but may be biassed by factors such as the location of the catch. Population estimate is 5000.

3	a i	1500	ii	120	b	100	c	1000

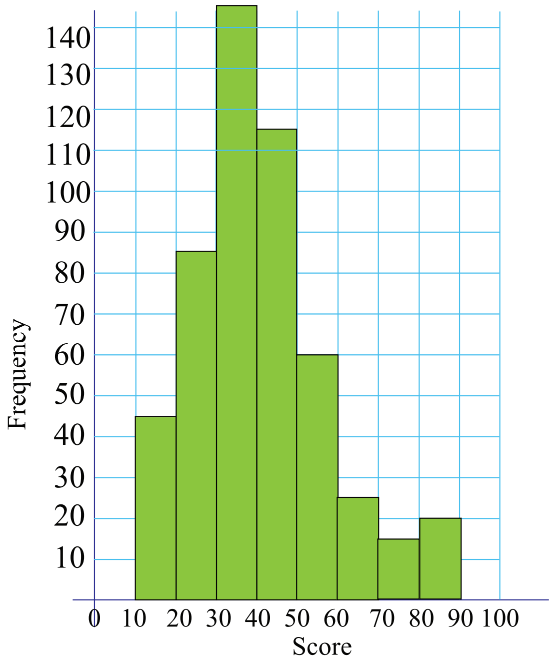
4 a, c numerical; b, d, e categorical

5 a, d discrete; b, c, e continuous

Exercise 5.1.2

1	a	53	b	34%	c	9.4%

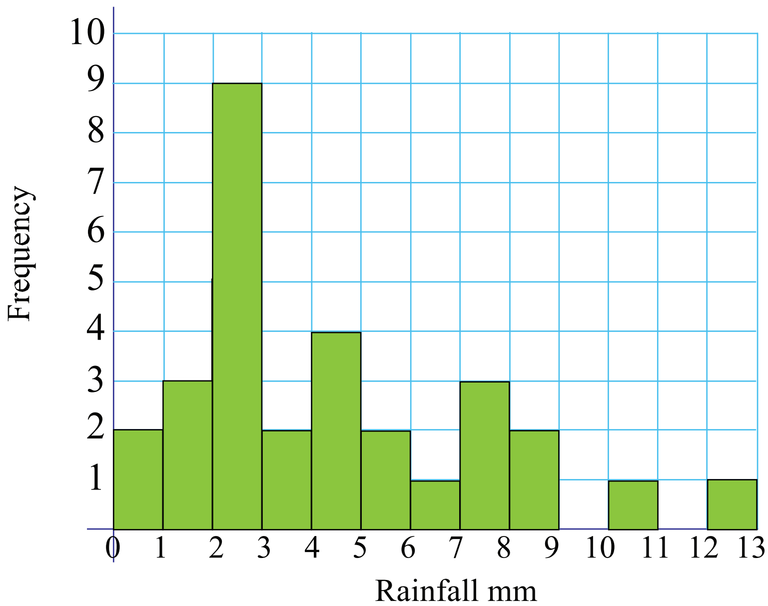
2



b 74% **c** 80

3 **a** Continuous

b



Exercise 5.1.3

1 Mode = 236–238 g; Mean = 234 g; Median = 235 g

2 Mode = 1.8–1.9 g; Mean = 1.69 g; Median = 1.80 g

3 Set A Mode = 29.1; Mean = 27.2; Median = 27.85

Set B Mode = 9; Mean = 26.6; Median = 9.

4 **a** \$27 522 **b** \$21 025 **c** Median

5 **a** \$233 300 **b** \$169 000 **c** Median

6 **a** 14.375 **b** 14.354

7 *b* A: 49.56 hr, B: 56.21 hr **c** Type B

8 12.22 cards

9 $a = 16$, $b = 3$

10 **b** 6010

Exercise 5.1.4

1 **a** Sample A Mean = 1.99 kg; Sample B Mean = 2.00 kg

b Sample A Sample std = 0.0552 kg; Sample B Sample std = 0.1877 kg

c Sample A Population std = 0.0547 kg; Sample B Population std = 0.1858 kg

2 **a** 16.4 **b** 6.83

3 Mean = 49.97; Std = 1.365

4 **a** \$84.67 **b** \$148

5 **a** 2.35 **b** 1.25

6 **a** \$232 **b** \$83

7 **c** 40

8 **a i** 20.17 **ii** 7.29 **b** 31 **c** 20.76

9 **a** 20 **b** $x + 1$

10 $\frac{y}{x+y}$

Exercise 5.2.1

1 **a** $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{2}{5}$

2 **a** $\frac{2}{7}$ **b** $\frac{5}{7}$

3 **a** $\frac{5}{26}$ **b** $\frac{21}{26}$

4 {HH, HT, TH, TT} **a** $\frac{1}{4}$ **b** $\frac{3}{4}$

5 {HHH,HHT,HTH,THH,TTT,TTH,THT,HTT} **a** $\frac{3}{8}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$

6 **a** $\frac{2}{9}$ **b** $\frac{2}{9}$ **c** $\frac{2}{3}$ **d** $\frac{1}{3}$

7 **a** $\frac{1}{2}$ **b** $\frac{3}{10}$ **c** $\frac{9}{20}$

8 **a** $\frac{11}{36}$ **b** $\frac{1}{18}$ **c** $\frac{1}{6}$ **d** $\frac{5}{36}$

9 {GGG, GGB. GBG, BGG, BBB, BBG, BGB, GBB} **a** $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{2}$

10 **a** $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{4}$

11 **a** $\frac{3}{8}$ **b** $\frac{1}{4}$ **c** $\frac{3}{8}$ **d** $\frac{3}{4}$

12 **a** {(1, H),(2, H),(3, H),(4,H),(5, H),(6, H),(1, T),(2, T),(3, T),(4, T),(5, T),(6,T)}
b $\frac{1}{4}$

13 **a** $\frac{1}{216}$ **b** $\frac{1}{8}$ **c** $\frac{3}{8}$

Exercise 5.3.1

1	**a**	$\frac{1}{4}$	**b**	$\frac{5}{8}$	**c**	$\frac{3}{4}$									
2	**a**	$\frac{1}{13}$	**b**	$\frac{1}{2}$	**c**	$\frac{1}{26}$		**d**	$\frac{7}{13}$						
3	$\frac{9}{26}$														
4	**a**	1.0	**b**	0.3	**c**	0.5									
5	**a**	0.65	**b**	0.70	**c**	0.65									
6	**a**	0.95	**b**	0.05	**c**	0.80									
7	**a** {TTT,TTH,THT,HTT,HHH,HHT,HTH,THH}						**b i**	$\frac{3}{8}$	**ii**	$\frac{1}{2}$	**iii**	$\frac{1}{4}$	**iv**	$\frac{3}{8}$	
8	**a**	$\frac{6}{25}$	**b**	$\frac{6}{25}$	**c**	$\frac{13}{25}$									
9	**b**	$\frac{3}{4}$	**c**	$\frac{1}{2}$	**d**	$\frac{1}{6}$	**e**	$\frac{7}{12}$							
10	**a**	$\frac{1}{4}$	**b**	$\frac{1}{2}$	**c**	$\frac{8}{13}$	**d**	$\frac{7}{13}$							
11	**a**	0.1399		**b i**	0.8797		**ii**	0.6							
12	**b**	$\frac{4}{15}$	**c**	$\frac{4}{15}$	**d**	$\frac{11}{15}$									

Exercise 5.4.1

15. Dale and Kritt are trying to solve a physics problem. The chances of solving the problem are Dale—65% and Kritt—75%. Find the probability that:

    a only Kritt solves the problem.

    b Kritt solves the problem.

    c both solve the problem.

    d Dale solves the problem given that the problem was solved.

16. A coin is weighted in such a way that there is a 70% chance of it landing heads. The coin is tossed three times in succession. Find the probability of observing:

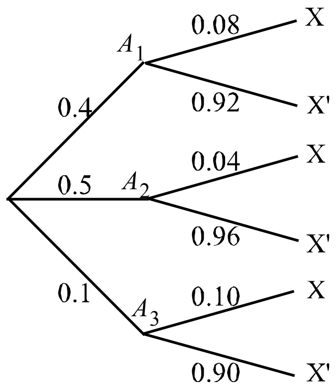
    a three tails.

    b two heads.

    c two heads given that at least one head showed up.

Exercise 5.4.2

11. For the tree diagram shown below, determine the probability $P(A_2|X)$.



12. Three factory employees, A, B and C, produce 40%, 30% and 30% of the total number of footballs in their division. Of these footballs, employee A produces 5% that are defective, employee B produces 6% that are defective and employee C produces 8% defectives.

    During an inspection round, a randomly selected football is found to be defective. What is the probability that employee A produced it?

13. Electrical components are checked for faults regularly at the CAMCO factory. A particular component is found to be non-defective 80% of the time, have a minor defect 12% of the time and a severe defect 8% of the time. Production levels for this component are such that 95% of the non-defective components are used for client X, 30% of the components that have a minor defect are used for client X and 5% of those that are severely defective will be used for client X.

    a Calculate the probability that a randomly selected component will be used for client X.

    b Calculate the probability that a component used for client X will have a severe defect.

14. WeCare Insurers have three types of motorcycle insurance policies, the low risk (L), moderate risk (M) and high risk (H) policies. The ratio of L to M to H policy holders is found to be 5:3:2. The respective probabilities of filing a claim by L, M and H policyholders is found to be 10%, 20% and 50% respectively.

    Calculate the probability that a policyholder who files a claim this year was a high-risk policyholder.

15. Machines M_1, M_2 and M_3 produce 35%, 45% and 20% of the total number of bolts produced at a steel factory. It is known that each machine produces defective items. The defective items are produced by M_1 in 2% of the time, by M_2 1% of the time and by M_3 3% of the time.

    A randomly chosen item is found to be defective. Which machine is most likely to have produced it?

16. Commuters arrive at a central station on three types of trains. Sixty per cent arrive using the Express, 30% are on the Fast train and the rest arrive using the Standard train. Of those commuters arriving on the Express, half are for business-related matters. Of those arriving on the Fast train, 60% are on business-related matters and, of those coming on the Standard, 90% are on business-related matters.

    Find the probability that a randomly selected commuter arriving at the station:

    a is travelling for business-related matters.

    b arrived using the Express train given that the person came for business-related matters.

Exercise 5.4.3

11. Eight people of different heights are to be seated in a row. The shortest and tallest in this group are not seated at either end. What is the probability that:

    a the tallest and shortest persons are sitting next to each other?

    b there is one person sitting between the tallest and shortest?

12. A committee of four is to be selected from a group of five boys and three girls. Find the probability that the committee consists of exactly two girls given that it contains at least one girl.

13. A bag contains 6 red marbles and 4 white marbles. Three marbles are randomly selected.

    a Find the probability that:

    b all three marbles are red.

    c all three marbles are red given that at least two of the marbles are red.

14. Four maths books, two chemistry books and three biology books are arranged in a row.

    a What is the probability that the books are grouped together in their subjects?

    b What is the probability that the chemistry books are not grouped?

15. A contestant on the game show "A Diamond for your Wife!" gets to select 5 diamonds from a box. The box contains 20 diamonds of which 8 are fakes.

    a Find the probability that the contestant will not bring a real diamond home for his wife.

    Regardless of how many real diamonds the contestant has after his selection, he can only take one home to his wife. A second contestant then gets to select from the remaining 15 diamonds in the box, but only gets to select one diamond.

    b What is the probability that this second contestant selects a real diamond?

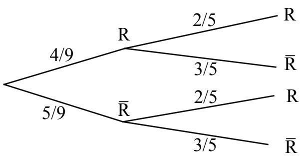
16. Light bulbs are sold in packs of 10. A quality inspector selects two bulbs at random without replacement. If both bulbs are defective the pack is rejected. If neither are defective the pack is accepted. If one of the bulbs is defective the inspector selects two more from the bulbs remaining in the pack and rejects the pack if one or both are defective. What are the chances that a pack containing 4 defective bulbs will in fact be accepted?

Exercise 5.4.1

1 **a** 0.7 **b** 0.75 **c** 0.50 **d** 0.5

2 **a** 0.5 **b** 0.83 **c** 0.10 **d** 0.90

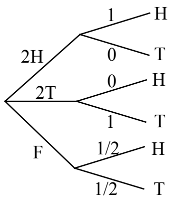
3 **a**



b $\frac{8}{45}$ **c** $\frac{22}{45}$ **d** $\frac{6}{11}$

4 **a** 0.5 **b** 0.30 **c** 0.25

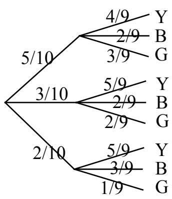
5 **a**



b $\frac{1}{2}$ **c** $\frac{2}{3}$

6 $\frac{1}{3}$

7 **a**



b $\frac{31}{45}$ **c** $\frac{2}{9}$

8 $\frac{2}{3}$

9 **a** 0.88 **b** 0.42 **c** 0.6 **d** 0.28

10 **a** 0.33 **b** 0.49 **c** 0.82 **d** 0.551

11 **a** 0.22 **b** 0.985 **c** 0.8629

12 **a** 0.44 **b** 0.733

14 **a** 0.512 **b** 0.128 **c** 0.8571

15 **a** 0.2625 **b** 0.75 **c** 0.4875 **d** 0.7123

16 **a** 0.027 **b** 0.441 **c** 0.453

Exercise 5.4.2

1 **a** 0.042 **b** 0.7143

2 **a** 0.4667 **b** 0.3868

3 **a** $\frac{5}{7}$ **b** $\frac{9}{13}$

4 $\frac{5}{9}$

5 **bi** $\frac{1}{40}$ **ii** 0.2

6 **ai** $\frac{2N-m}{2N}$ **ii** $\frac{2(N-m)}{2N-m}$ **b** $\frac{m}{m+(N-m)2^n}$

7 $\frac{9}{19}$

8 **a** 0.07 **b** 0.3429 **c** 0.30 **d** 0.0282

9 **a** 0.8008 **b** 0.9767 **c** 0.0003

10 **a** 0.0464 **b** 0.5819 **c** 0.9969

11 $\frac{1}{31}$

12 $\frac{10}{31}$

13 **a** 0.8 **b** 0.005

14 $\frac{10}{21}$

15 M_1

16 **a** 0.57 **b** $\frac{18}{57}$

Exercise 5.4.3

1 **a** $\frac{5}{126}$ **b** $\frac{5}{18}$ **c** $\frac{1}{126}$

2 **a** $\frac{1}{5}$ **b** $\frac{1}{10}$ **c** $\frac{2}{5}$ **d** $\frac{3}{5}$

3 **a** $\frac{72}{5525}$ **b** $\frac{1}{5525}$ **c** $\frac{1}{1201}$

4 $\frac{2}{5}$

5 **a** $\frac{63}{143}$ **b** $\frac{133}{143}$

6 **a** $\frac{5}{12}$ **b** $\frac{5}{33}$ **c** $\frac{5}{6}$

7 $\frac{3}{11}$

8 **a** $\frac{4}{13}$ **b** $\frac{9}{13}$

9 **a** $\frac{67}{91}$ **b** $\frac{22}{91}$

10 **a** $\frac{1}{4}$ **b** $\frac{1}{28}$ **c** $\frac{5}{14}$

11 **a** $\frac{5}{28}$ **b** $\frac{1}{28}$

12 $\frac{6}{13}$

13 **a** $\frac{1}{6}$ **b** $\frac{1}{4}$

14 **a** $\frac{1}{210}$ **b** $\frac{7}{9}$

15 **a** $\frac{7}{1938}$ **b** 0.6

16 $\frac{11}{21}$

Exercise 5.5.1

11. A box contains four balls numbered 1 to 4. A ball is selected at random from the box and its number is noted.

    a If the random variable X denotes the number on the ball, find the probability distribution of X.

    After the ball is placed back into the box, a second ball is randomly selected.

    b If the random variable S denotes the sum of the numbers shown on the balls after the second draw, find the probability distribution of S.

12. A probability distribution function for the random variable X is defined by:

    $P(X = x) = k \times (0.9)^x, x = 0, 1, 2, \ldots$

    Find: **a** $P(X \geq 2)$.

    b $P(1 \leq X < 4)$.

Exercise 5.5.2

16 A game is played by selecting coloured discs from a box. The box initially contains two red and eight blue discs. Tom pays $10.00 to participate in the game. Each time Tom participates he selects two discs. The winnings are governed by the probability distribution shown below, where the random variable N is the number of red discs selected.

n Winnings	0 $0	1 $W	2 $5W
$P(N = n)$			

a Complete the table.

b For what value of W will the game be fair?

17. A random variable X has the following probability distribution:

x	0	1	2
$P(X = x)$	a	$\frac{1}{3}(1-b)$	$\frac{1}{3}b$

a What values may a and b take?

b Express, in terms of a and b : **i** $E(X)$ **ii** $Var(X)$.

18. a Find the mean and variance of the probability distribution defined by:

$P(Z = z) = k(0.8)^z, z = 0, 1, 2, \ldots$

bi Show $P(X = x) = p \times (1-p)^x, x = 0, 1, 2, \ldots$ defines a probability distribution.

ii Show $E(X) = \frac{1-p}{p}$.

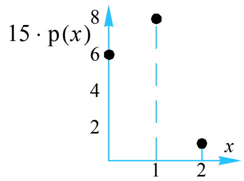
iii Show $Var(X) = \frac{1-p}{p^2}$.

Exercise 5.5.1

1 0.3

2 **a** 0.1 **bi** 0.2 **ii** 0.7

3 **a** $p(0) = \frac{6}{15}, p(1) = \frac{8}{15}, p(2) = \frac{1}{15}$ **b**



c $\frac{14}{15}$

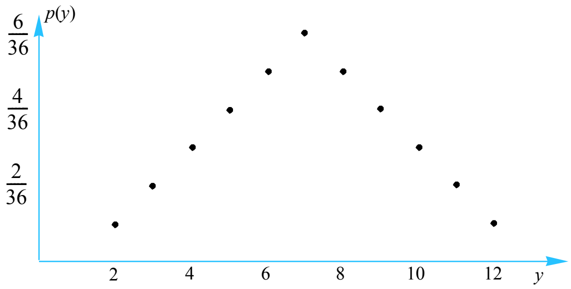
4 **a** {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

b

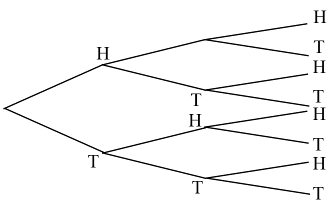
x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c $\frac{5}{36}$

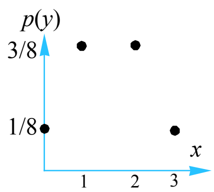
d



5 **a**



b



$p(0) = \frac{1}{8}$

$p(1) = \frac{3}{8}$

$p(2) = \frac{3}{8}$

$p(3) = \frac{1}{8}$

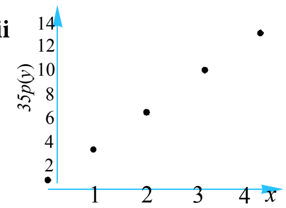
c $\frac{4}{7}$

6 **a** $\frac{1}{35}$ **bi** $p(0) = \frac{1}{35}$ $p(1) = \frac{4}{35}$

$p(2) = \frac{7}{35}$ $p(3) = \frac{10}{35}$

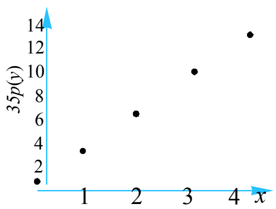
$p(4) = \frac{13}{35}$

ii



c $\frac{6}{7}$

7



a i 0.9048 **ii** 0.09048 **b** 0.0002

8 0.3712

9 **a** $p(0) = \frac{11}{30}, p(-1) = \frac{1}{2}, p(3) = \frac{2}{15}$ **b i** $\frac{11}{30}$ **ii** $\frac{13}{15}$

10

n	0	1	2
P(N = n)	$\frac{6}{15}$	$\frac{8}{15}$	$\frac{1}{15}$

11 **a**

n	1	2	3	4
$P(N = n)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b

s	2	3	4	5	6	7	8
P(S = s)	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

12 **a** 0.81 **b** 0.2439

Exercise 5.5.2

1 **a** 2.8 **b** 1.86

2 **a** 3 **b i** 1 **ii** 1 **c i** 6 **ii** 0.4

3 **a i** 1.3 **ii** 2.5 **iii** –0.1

b i 0.9 **ii** 7.29 **c i** $\frac{31}{60}$ **ii** 0.3222

4 $\mu = \frac{2}{3}, \sigma^2 = 0.3556$

5 **a** 7 **b** 5.8333

6 $np = 3 \times \frac{1}{2} = 1.5$

7 **a** $\frac{1}{25}$ **b** 2.8 **c** 1.166

8 **a** 0.1 **b i** 0.3 **ii** 1

c i 0 **ii** 1 **iii** 2

9 5.56

10 $p(0) = \frac{35}{120}, p(1) = \frac{63}{120}, p(2) = \frac{21}{120}, p(3) = \frac{1}{120}$ **b i** 0.9 **ii** 0.49

c $W = 3N - 3$, $E(W) = -0.3$

11 **a** \$ –1.00 **b** both the same

12 **a** 50 **b** 18 **c** 2

13 **a** 11 **b** $\frac{\sqrt{3}}{3}$ **c** −4

14 **a** 0.75 **b** 0.6339

15 **a** $E(X) = 1 - 2p$, $Var(X) = 4p(1 - p)$ **b** **i** $n(1 - 2p)$ **ii** $4np(1 - p)$

16 **a**

n	0	1	2
P(N = n)	$\frac{28}{45}$	$\frac{16}{45}$	$\frac{1}{45}$

b $W = 21.43$

17 **a** $a = \frac{2}{3}, 0 \le b \le 1$ **b** $E(X) = \frac{b+1}{3}$, $Var(X) = \frac{1}{9}(2 + 7b - b^2)$

18 **a** $E(X) = 4$, $Var(X) = 20$

Exercise 5.6.1

16. In a suburb, it is known that 40% of the population are blue-collar workers. A delegation of one hundred volunteers are each asked to sample 10 people in order to determine if they are blue-collar workers. The town has been divided into 100 regions so that there is no possibility of doubling up (i.e. each worker is allocated one region). How many of these volunteers would you expect to report that there were fewer than 4 blue-collar workers?

17. Show that if $X \sim B(n, p)$, then:

$$P(X = x + 1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{1-p}\right)P(X = x)\ ,\ x = 0, 1, 2, \ldots, n-1$$

18. Show that if $X \sim B(n, p)$, then:

a $E(X) = np$.   b $Var(X) = np(1-p)$.

19. Mifumi has ten pots labelled one to ten. Each pot and its content can be considered to be identical in every way. Mifumi plants a seed in each pot, such that each seed has a germinating probability of 0.8.

a Find the probability that:

i all the seeds will germinate.

ii exactly three seeds will germinate.

iii more than eight seeds germinate.

b How many pots must Mifumi use to be 99.99% sure to obtain at least one flower?

20. A fair die is rolled eight times. If the random variable X denotes the number of fives observed, find:

a $E(X)$.   **b** $Var(X)$.   **c** $E\left(\frac{1}{8}X\right)$.   **d** $Var\left(\frac{1}{8}X\right)$.

21. A bag contains 5 balls of which 2 are red. A ball is selected at random. Its colour is noted and then it is replaced in the bag. This process is carried out 50 times. Find:

a the mean number of red balls selected.

b the standard deviation of the number of red balls selected.

22. The random variable X is $B(n, p)$ distributed such that $\mu = 9$ and $\sigma^2 = 3.6$. Find:

a $E(X^2 + 2X)$.   **b** $P(X = 2)$.

23. a If $X \sim Bin(10, 0.6)$, find: **i** $E(X)$. **ii** $Var(X)$.

b If $X \sim Bin(15, 0.4)$, find: **i** $E(X)$. **ii** $Var(X)$.

24. The random variable X has a binomial distribution such that $E(X) = 12$ and $Var(X) = 4.8$. Find $P(X = 12)$.

25. Metallic parts produced by an automated machine have some variation in their size. If the size exceeds a set threshold, the part is labelled as defective. The probability that a part is defective is 0.08. A random sample of 20 parts is taken from the day's production. If X denotes the number of defective parts in the sample, find its mean and variance.

26. Quality control for the manufacturing of bolts is carried out by taking a random sample of 15 bolts from a batch of 10,000. Empirical data shows that 10% of bolts are found to be defective. If three or more defectives are found in the sample, that particular batch is rejected.

    a Find the probability that a batch is rejected.

    b The cost to process the batch of 10,000 bolts is \$20.00. Each batch is then sold for \$38.00, or it is sold as scrap for \$5.00 if the batch is rejected. Find the expected profit per batch.

27. In a shooting competition, a competitor knows (that on average) she will hit the bulls-eye on three out of every five attempts. If the competitor hits the bulls-eye she receives \$10.00.

    However, if the competitor misses the bulls-eye but still hits the target region she only receives \$5.00.

    a What can the competitor expect in winnings on any one attempt at the target?

    b How much can the competitor expect to win after 20 attempts?

28. A company manufactures bolts which are packed in batches of 10,000. The manufacturer operates a simple sampling scheme whereby a random sample of 10 is taken from each batch. If the manufacturer finds that there are fewer than 3 faulty bolts the batch is allowed to be shipped out. Otherwise, the whole batch is rejected and reprocessed.

    a If 10% of all bolts produced are known to be defective, find the proportion of batches that will be reprocessed.

    b Show that if $100p$% of bolts are known to be defective, then P(Batch is accepted) = $(1-p)^8(1+8p+36p^2)$, $0 \le p \le 1$

    c Using a graphics calculator, sketch the graph of P('Batch is accepted') versus p.

    Describe the behaviour of this curve.

29. Large batches of screws are produced by TWIST'N'TURN Manufacturers Ltd. Each batch consists of N screws and has a proportion p of defectives. It is decided to carry out an inspection of the product, by selecting 4 screws at random and accepting the batch if there is no more than one defective, otherwise the batch is rejected.

    a Show that P(Accepting any batch) = $(1-p)^3(1+3p)$.

    b Sketch a graph showing the relationship between the probability of accepting a batch and p (the proportion of defectives).

30. A quality control process for a particular electrical item is set up as follows:

    A random sample of 20 items is selected. If there is no more than one faulty item the whole batch is accepted. If there are more than two faulty items the batch is rejected. If there are exactly two faulty items, a second sample of 20 items is selected from the same batch and is accepted only if this second sample contains no defective items.

    Let p be the proportion of defectives in a batch.

    a Show that the probability, $\Phi(p)$, that a batch is accepted is given by:

$$\Phi(p) = (1-p)^{19}[1+19p+190p^2(1-p)^{19}],\ 0 \le p \le 1 .$$

b Find the probability of accepting this batch if it is known that 5% of all items are defective.

c If 200 such batches are produced each day, find an estimate of the number of batches that can be expected to be rejected on any one day.

Challenging question!

31. Given that the random variable X denotes the number of successes in n Bernoulli trials, with probability of success on any given trial represented by p:

a find $E(X|X \geq 2)$. **b** show that $\sigma \leq \frac{1}{2}\sqrt{n}$.

Exercise 5.6.2

14. On average, it is found that 8 out of every 10 electric components produced from a large batch have at least one defective component. Find the probability that there will be at least 2 defective components from a randomly selected batch.

15. Flaws, called seeds, in a particular type of glass sheet occur at a rate of 0.05 per square metre. Find the probability that a rectangular glass sheet measuring 4 metres by 5 metres contains:

a no seeds. **b** at least two seeds.

Sheets containing at least two seeds are rejected.

c Find the probability that, in a batch of ten such glass sheets, at most one is rejected.

16. Simar has decided to set up a small business venture. The venture requires Simar to go fishing every Sunday so he can sell his catch on the Monday. He realises that on a proportion p of these days he does not catch anything.

a Find the probability that on any given Sunday, Simar catches:

i no fish. **ii** one fish. **iii** at least two fish.

The cost to Simar on any given Sunday if he catches no fish is \$5. If he catches one fish Simar makes a profit of \$2 and if he catches more than one fish he makes a profit of \$10. Let the random variable X denote the profit Simar makes on any given Sunday.

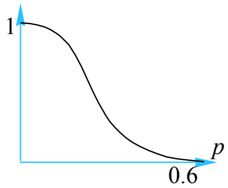
b Show that $E(X) = 10 - 15p + 8p\ln p$, $0 < p < 1$.

c Find the maximum value of p, if Simar is to make a positive gain on his venture.

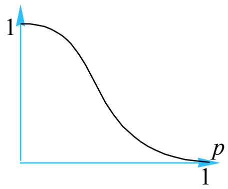
Exercise 5.6.1

1	a	0.2322	b	0.1737	c	0.5941		
2	a	0.3292	b	0.8683	c	0.2099	d	0.1317
3	a	0.1526	b	0.4812	c	0.5678		
4	a	0.7738	b	3.125×10^{-7}	c	0.9988	d	3×10^{-5}
5	a	0.2787	b	0.4059				
6	a	0.2610	b	0.9923				
7	a	0.2786	b	0.7064	c	0.1061		
8	a	0.1318	b	0.8484	c	0.054	d	0.326
9	a	0.238	b	0.6531	c	0.0027	d	0.726
	e	12.86						
10	a	0.003	b	0.2734	c	0.6367	d	0.648
11	a	0.3125	b	0.0156	c	0.3438	d	3
12	a	0.2785	b	0.3417	c	120		
13	a	0.0331	b	0.565				
14	a	0.4305	b	0.61	c	$720	d	0.2059
15	a i	1.4	ii	1	iii	1.058	iv	0.0795
	v	0.0047						
	b i	3.04	ii	3	iii	1.373	iv	0.2670
	v	0.1390						
16	38.23							
19	a i	0.1074	ii	7.9×10^{-4}	iii	0.3758	b	at least 6
20	a	$\frac{4}{3}$	b	$\frac{10}{9}$	c	$\frac{1}{6}$	d	$\frac{5}{288}$
21	a	20	b	3.4641				
22	a	102.6	b	0.000254				
23	a i	6	ii	2.4	b i	6	ii	3.6
24	0.1797							
25	1.6, 1.472							
26	**a** 0.1841 **b** $11.93							
27	**a** $8 **b** $160							

28 **a** 0.0702 **c**



29 **b**



30 **b** 0.8035 **c** 39.3

31 **a** $\dfrac{np - np(1-p)^{n-1}}{1-(1-p)^n - np(1-p)^{n-1}}, 0 < p < 1$

Exercise 5.6.2

1 **a** $P(X = x) = \dfrac{e^{-2}2^x}{x!}, x = 0, 1, 2, \ldots$

b i 0.1353 **ii** 0.2707 **iii** 0.5940 **iv** 0.4557

2 **a** 0.0383 **b** 0.1954

3 **a** 0.2052 **b** 0.9179

4 **a** 0.2623 **b** 0.8454

5 **a** 0.0265 **b** 0.0007

6 **a** 0.1889 **b** 0.7127

7 **a** 0.7981 **b** 0.2019 **c** 0.1835

8 **a** 0.2661 **b** 0.5221

9 0.1912

10 **a** 0.3504 **b** 0.6817

11 **a** 0.00127 **b** 0.0500

12 **a** 0.1804 **b** 0.0166 **c** 0.3233

13 **a** 0.8131; 0.5511 No

14 14. 0.4781

15 **a** 0.3679 **b** 0.2642 **c** 0.2135

16 **a i** p **ii** $-p\ln p$ **iii** $1 - p + p\ln p$ **c** 0.4785

Exercise 5.7.2

1. If Z is a standard normal random variable, find:

   c $p(Z \geq 0.5)$ d $p(Z \leq 1.2)$ e $p(Z \geq 1.5)$ f $p(Z \leq 2)$

2. If Z is a standard normal random variable, find:

   c $p(Z \geq -0.5)$ d $p(Z \leq -1.2)$ e $p(Z \geq -1.5)$ f $p(Z \leq -2)$

3. If Z is a standard normal random variable, find:

   c $p(1.5 \leq Z < 2.1)$

4. If Z is a standard normal random variable, find:

   c $p(-1.5 \leq Z < -0.1)$

5. If X is a normal random variable with mean $\mu = 8$ and variance $\sigma^2 = 4$, find:

   c $p(X < 9.5)$

6. If X is a normal random variable with mean $\mu = 100$ and variance $\sigma^2 = 25$, find:

   c $p(X < 95)$

7. If X is a normal random variable with mean $\mu = 60$ and standard deviation $\sigma = 5$, find:

   c $p(50 \leq X < 55)$

17. For a normal variable, X, $\mu = 196$ and $\sigma = 4.2$. Find:

   c $p(193.68 < X < 196.44)$

32. At a Junior track and field meet it is found that the times taken for children aged 14 to sprint the 100 metres race are normally distributed with a mean of 15.6 seconds and standard deviation of 0.24 seconds. Find the probability that the time taken for a 14 year old at the meet to sprint the 100 metres is:

   i less than 15 seconds.

   ii at least 16 seconds.

   iii between 15 and 16 seconds.

   On one of the qualifying events, eight children are racing. What is the probability that six of them will take between 15 and 16 seconds to sprint the 100 metres?

33. Rods are manufactured to measure 8 cm. Experience shows that these rods are normally distributed with a mean length of 8.02 cm and a standard deviation of 0.04 cm.

   Each rod costs $5.00 to make and is sold immediately if its length lies between 8.00 cm and 8.04 cm. If its length exceeds 8.04 cm it costs an extra $1.50 to reduce its length to 8.02 cm. If its length is less than 8.00 cm it is sold as scrap metal for $1.00.

   a What is the average cost per rod? b What is the average cost per usable rod?

34. The resistance of heating elements produced is normally distributed with mean 50 ohms and standard deviation 4 ohms.

a Find the probability that a randomly selected element has resistance less than 40 ohms.

bi If specifications require that acceptable elements have a resistance between 45 and 55 ohms, find the probability that a randomly selected element satisfies these specifications.

ii A batch containing 10 such elements is tested. What is the probability that exactly 5 of the elements satisfy the specifications?

c The profit on an acceptable element, i.e. one that satisfies the specifications, is \$2.00, while unacceptable elements result in a loss of \$0.50 per element. If \$ P is the profit on a randomly selected element, find the profit made after producing 1000 elements.

35.

a Find the mean and standard deviation of the normal random variable X, given that $P(X < 50) = 0.05$ and $P(X > 80) = 0.1$.

b Electrical components are mass-produced and have a measure of 'durability' that is normally distributed with mean μ and standard deviation 0.5.

The value of μ can be adjusted at the control room. If the measure of durability of an item scores less than 5, it is classified as defective. Revenue from sales of non-defective items is \$ S per item, while revenue from defective items is set at \$$^1/_{10}S$. Production cost for these components is set at \$$^1/_{10}\mu S$. What is the expected profit per item when μ is set at 6?

36. From one hundred first year students sitting the end-of-year Botanical Studies 101 exam, 46 of them passed while 9 were awarded a high distinction.

a Assuming that the students' scores were normally distributed, determine the mean and variance on this exam if the pass mark was 40 and the minimum score for a high distinction was 75.

Some of the students who failed this exam were allowed to sit a 'make-up' exam in early January of the following year. Of those who failed, the top 50% were allowed to sit the 'make-up' exam.

b What is the lowest possible score that a student can be awarded in order to qualify for the 'make-up' exam.

37. The heights of men in a particular country are found to be normally distributed with mean 178 cm and a standard deviation of 5 cm. A man is selected at random from this population.

a Find the probability that this person is:

i at least 180 cm tall

ii between 177 cm and 180 cm tall.

b Given that the person is at least 180 cm, find the probability that he is:

i at least 184 cm

ii no taller than 182 cm.

c If ten such men are randomly selected, what are the chances that at least two of them are at least 176 cm?

Exercise 5.7.1

1	**a**	0.6915	**b**	0.9671	**c**	0.9474	**d**	0.9965
	e	0.9756	**f**	0.0054				
2	**a**	0.0360	**b**	0.3759	**c**	0.0623	**d**	0.0564
	e	0.0111						

Exercise 5.7.2

1	**a**	0.0228	**b**	0.9332	**c**	0.3085	**d**	0.8849
	e	0.0668	**f**	0.9772				
2	**a**	0.9772	**b**	0.0668	**c**	0.6915	**d**	0.1151
	e	0.9332	**f**	0.0228				
3	**a**	0.3413	**b**	0.1359	**c**	0.0489		
4	**a**	0.6827	**b**	0.1359	**c**	0.3934		
5	**a**	0.8413	**b**	0.4332	**c**	0.7734		
6	**a**	0.1151	**b**	0.1039	**c**	0.1587		
7	**a**	0.1587	**b**	0.6827	**c**	0.1359		
8	**a**	0.1908	**b**	0.4754	**c**	16.88		
9	**a**	0.1434	**b**	0.6595				
10	**a**	0.2425	**b**	0.8413	**c**	0.5050		
11	**a**	−1.2816	**b**	0.2533				
12	**a**	58.2243	**b**	41.7757	**c**	59.80		
13	39.11							
14	9.1660							
15	42%							
16	0.7021							
17	**a**	0.2903	**b**	0.4583	**c**	0.2514		
18	23%							
19	0.5							
20	11%							
21	5%							
22	14%							
23	1.8							

24	252								
25	0.1517								
26	0.3821								
27	0.22								
28	322								
29	0.1545								
30	7								
31	87								
32	**a i**	0.0062	**ii**	0.0478	**iii**	0.9460	**b**	0.0585
33	**a**	\$5.11	**b**	\$7.39				
34	**a**	0.0062	**b i**	0.7887	**ii**	0.0324	**c**	\$1472
35	**a**	$\mu = 66.86, \sigma = 10.25$			**b**	\$0.38*S*		
36	**a**	$\mu = 37.2, \sigma = 28.2$			**b**	20 (19.9)		
37	**a i**	0.3446	**ii**	0.2347	**b i**	0.3339	**ii**	0.3852
	c	0.9995						

Example 6.1.4

The concentration of a drug, in milligrams per millilitre, in a patient's bloodstream, t hours after an injection, is approximately modelled by the function:

$$t \mapsto \frac{2t}{8+t^2}, t \geq 0$$

Find the average rate of change in the concentration of the drug present in a patient's bloodstream:

a during the first hour

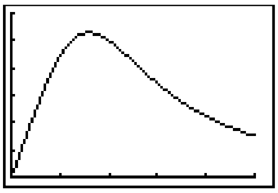
b during the first two hours

c during the period $t = 2$ to $t = 4$.

To help us visualise the behaviour of this function we will make use of the TI–83.

Begin by introducing the variable C, to denote the concentration of the drug in the patient's bloodstream t hours after it is administered.

So that $C(t) = \frac{2t}{8+t^3}, t \geq 0$.



Initially the concentration is 0 milligrams per millilitre. The concentration after 1 hour is given by $C(1) = \frac{2 \times 1}{8 + 1^3} = \frac{2}{9} \approx 0.22$.

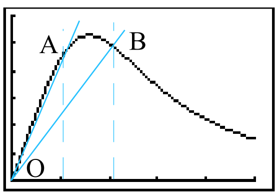
Therefore, the average rate of change in concentration (C_{ave}) during the first hour is given by $C_{ave} = \frac{0.22 - 0}{1 - 0} = 0.22$. Note: the units are *mg/mL/hr*.

The concentration 2 hours after the drug has been administered is $C(2) = \frac{2 \times 2}{8 + 2^3} = 0.25$. That is, 0.25 *mg/ml*.

Therefore, the average rate of change in concentration with respect to time is: $C_{ave} = \frac{0.25 - 0}{2 - 0} = 0.125$.

Notice that although the concentration has increased (compared to the concentration after 1 hour), the rate of change in the concentration has actually decreased!

This should be evident from the graph of $C(t)$ versus t.



The slope of the straight line from the origin to A(1, 0.22), m_{OA}, is greater than the slope from the origin O to the point B(2, 0.25), m_{OB}.

That is $m_{OA} > m_{OB}$.

The average rate of change in concentration from $t = 2$ to $t = 4$ is given by $\dfrac{C(4) - C(2)}{4 - 2}$.

Now, $\dfrac{C(4) - C(2)}{4 - 2} = \dfrac{\dfrac{2 \times 4}{8 + 4^3} - 0.250}{4 - 2} \approx \dfrac{0.111 - 0.250}{2} = -0.0694$

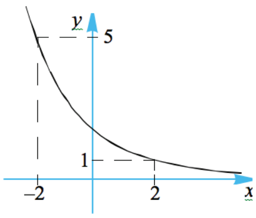
Therefore, the average rate of change of concentration is -0.070 *mg/ml/hr*,

i.e. the overall amount of drug in the patient's bloodstream is decreasing during the time interval $2 \le t \le 4$.

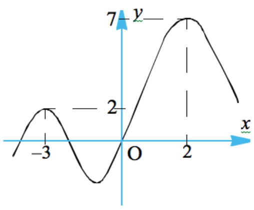
Exercise 6.1.1

1. For each of the following graphs determine the average rate of change over the specified domain.

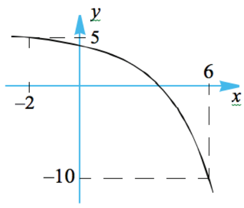
c $x \in [-2, 2]$



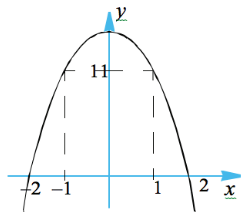
d $x \in [-3, 2]$



e $x \in [-2, 6]$



f $x \in [-1, 1]$



8. For the case where r is 20 cm,

a find the average rate of increase in the amount of water inside the bowl with respect to its height, h cm, as the water level rises from 2 cm to 5 cm.

b Find the average rate of increase in the amount of water inside the bowl with respect to its height, h cm, as the water level rises by

i 1 cm ii 0.1 cm iii 0.01 cm.

9. An amount of money is placed in a bank and is accumulating interest on a daily basis. The table below shows the amount of money in the savings account over a period of 600 days.

t (days)	100	200	300	400	500	600	700
\$$D$/day	1600	1709	1823	1942	2065	2194	2328

a Plot the graph of \$$D$ versus t (days).

b Find the average rate of change in the amount in the account during the period of 100 days to 300 days.

10. The temperature of coffee since it was poured into a cup was recorded and tabulated below.

***t* min**	0	2	4	6	9
***T* °C**	60	50	30	10	5

a Plot these points on a set of axes that show the relationship between the temperature of the coffee and the time it has been left in the cup.

b Find the average rate of change of temperature of the coffee over the first 4 minutes.

c Over what period of time is the coffee cooling the most rapidly?

11. The displacement, *d* metres, of an object, *t* seconds after it was set in motion is described by the equation:

$$d = 4t + 5t^2 \text{, where } t \geq 0.$$

a Find the distance that the object travels in the first 2 seconds of its motion.

b Find the average rate of change of distance with respect to time undergone by the object over the first 2 seconds of its motion.

c What quantity is being measured when determining the average rate of change of distance with respect to time?

d How far does the object travel during the 5th second of motion?

e Find the object's average speed during the 5th second.

12. A person invests \$1000 and estimates that, on average, the investment will increase each year by 16% of its value at the beginning of the year.

a Calculate the value of the investment at the end of each of the first 5 years.

b Find the average rate at which the investment has grown over the first 5 years.

Exercise 6.1.2

4. For each of the functions, f, given below, find the gradient of the secant joining the points $P(a, f(a))$ and $Q(a+h, f(a+h))$ and hence deduce the gradient of the tangent drawn at the point P.

   a $f(x) = x$ b $f(x) = x^2$

   c $f(x) = x^3$ d $f(x) = x^4$.

   Hence deduce the gradient of the tangent drawn at the point $P(a, f(a))$ for the function $f(x) = x^n, n \in N$.

6. The healing process of a certain type of wound is measured by the decrease in surface area that the wound occupies on the skin. A certain skin wound has its surface area modelled by the equation $S(t) = 20 \times 2^{-0.1t}$ where S sq. cm is the unhealed area t days after the skin received the wound.

   a Sketch the graph of $S(t) = 20 \times 2^{-0.1t}, t \geq 0$.

   b i What area did the wound originally cover?

   ii What area will the wound occupy after 2 days?

   iii How much of the wound healed over the two day period?

   iv Find the average rate at which the wound heals over the first two days.

   c How much of the wound would heal over a period of h days?

   d Find the rate at which the wound heals:

   i immediately after it occurs

   ii one day after it occurred.

Exercise 6.1.1

1	**a**	$\frac{3}{4}$	**b**	$\frac{3a}{4b}$	**c**	-1	**d**	1
	e	$-\frac{15}{8}$	**f**	0				
2	**a**	4	**b**	0.2	**c**	0.027	**d**	0.433
	e	-0.01	**f**	6.34	**g**	6.2	**h**	0
3	**a**	6 m/s	**b**	30 m/s	**c**	$11 + 6h + h^2$ m/s		

4 12 m/s

5 $8 + 2h$

6 -3.49°C/sec

7 **a** 127π cm³/cm

b i 19.6667π cm³/cm **ii** 1.9967π cm³/cm **iii** 0.2000π cm³/cm

8 1.115

9 **a** -7.5°C/min **b** $t = 2$ to $t = 6$

10 **a** 28 m **b** 14 m/s **c** average speed

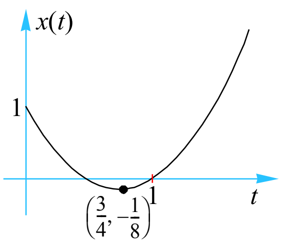
d 49 m **e** 49 m/s

11 **a** \$1160, \$1345.6, \$1560.90, \$1810.64, \$2100.34 **b** \$220.07 per year

Exercise 6.1.2

1	**a**	$h + 2$	**b**	$4 + h$	**c**	$\frac{-1}{1+h}$	**d**	$3 - 3h + h^2$
2	**a**	2	**b**	4	**c**	-1	**d**	3
3	**a**	$2a + h$	**b**	$-(2a + h)$	**c**	$(2a + 2) + h$	**d**	$3a^2 + 1 + 3ah + h^2$
	e	$-(3a^2 + 3ah + h^2)$			**f**	$3a^2 - 2a + (3a - 1)h + h^2$		
	g	$\frac{-2}{a(a+h)}$	**h**	$-\frac{1}{(a-1)(a-1+h)}$			**i**	$\frac{1}{\sqrt{a+h}+\sqrt{a}}$

4 **a**

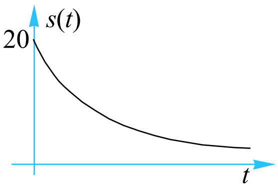


b i 3 ms^{-1} **ii** 2 ms^{-1} **iii** 1.2 ms^{-1}

d Find (limit) as $h \to 0$ **e** $4t - 3$

5 **a** 1 ; 1 **b** $2a + h$; $2a$ **c** $3a^2 + 3ah + h^2$; $3a^2$ **d** $4a^3 + 6a^2h + 4ah^2 + h^3$; $4a^3$

6 **a**



b i 20 cm² **ii** 17.41 cm² **iii** 2.59

Exercise 6.1.3

1 **a** 3 **b** 8 **c** $-\frac{1}{9}$ **d** 1.39

e -1 **f** $\frac{17}{16}$

2 **a** 4.9 m **b** $4.9(h^2 + 2h)$ m **c** 9.8 m/s

3 **a** $8x$ **b** $10x$ **c** $12x^2$ **d** $15x^2$

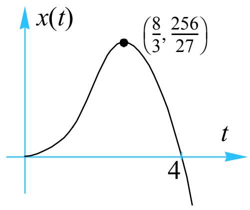
e $16x^3$ **f** $20x^3$

4 **a** $4x$ **b** -1 **c** $-1 + 3x^2$ **d** $-x^{-2}$

e $-2(x+1)^{-2}$ **f** $0.5x^{-1/2}$

5 **a** 1 ms^{-1} **b** $(2 - a)$ ms^{-1}

6 **a**



b i 5 ms^{-1} **ii** 4 ms^{-1} **c** $8t - 3t^2$ ms^{-1} **d** $\frac{8}{3}$ sec

Exercise 6.2.2

2. Differentiate the following with respect to the independent variable:

a $v = \frac{2}{3}\left(5 - \frac{2}{t^2}\right)$

b $S = \pi r^2 + \frac{20}{r}$

c $q = \sqrt{s^5} - \frac{3}{s}$

d $h = \frac{2 - t + t^2}{t^3}$

e $L = \frac{4 - \sqrt{b}}{b}$

f $W = (m-2)^2(m+2)$

Exercise 6.2.3

3. Differentiate the following.

g $\frac{4}{x^2} \times \sin x$ h $xe^x \sin x$ i $xe^x \log_e x$

4. Differentiate the following.

g $\frac{e^x - 1}{x + 1}$ h $\frac{\sin x + \cos x}{\sin x - \cos x}$ i $\frac{x^2}{x + \log_e x}$

5. Differentiate the following.

f $\cos(-4x) - e^{-3x}$ g $\log_e(4x + 1) - x$ h $\log_e(e^{-x}) + x$

i $\sin\left(\frac{x}{2}\right) + \cos(2x)$ j $\sin(7x - 2)$ k $\sqrt{x} - \log_e(9x)$

l $\log_e(5x) - \cos(6x)$

6. Differentiate the following.

i $\cos(\sin\theta)$ j $4\sec\theta$ k $\operatorname{cosec}(5x)$ l $3\cot(2x)$

7. Differentiate the following.

k $e^{-\cos(2\theta)}$ l $e^{2\log_e(x)}$ m $\frac{2}{e^{-x} + 1}$

n $(e^x - e^{-x})^3$ o $\sqrt{e^{2x + 4}}$ p $e^{-x^2 + 9x - 2}$

8. Differentiate the following.

i $\log_e\left(\frac{1}{\sqrt{x + 2}}\right)$ j $\log_e(\cos^2 x + 1)$ k $\log_e(x\sin x)$ l $\log_e\left(\frac{x}{\cos x}\right)$

9. Differentiate the following.

i $\frac{\cos(2x)}{e^{1 - x}}$ j $x^2 \log_e(\sin 4x)$ k $e^{-\sqrt{x}} \sin\sqrt{x}$ l $\cos(2x\sin x)$

m $\frac{e^{5x + 2}}{1 - 4x}$ n $\frac{\log_e(\sin\theta)}{\cos\theta}$ o $\frac{x}{\sqrt{x + 1}}$ p $x\sqrt{x^2 + 2}$

q $(x^3 + x)^3\sqrt[3]{x + 1}$ r $(x^3 - 1)\sqrt{x^3 + 1}$ s $\frac{1}{x}\log_e(x^2 + 1)$

t $\log_e\left(\frac{x^2}{x^2 + 2x}\right)$ u $\frac{\sqrt{x - 1}}{x}$ v $e^{-x}\sqrt{x^2 + 9}$

w $(8-x^3)\sqrt{2-x}$ x $x^n \ln(x^n-1)$

15. Find: c $\frac{d}{dx}(\cos x°)$

17. a Given that $f(x) = 1-x^3$ and $g(x) = \log_e x$, find: i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$

b Given that $f(x) = \sin(x^2)$ and $g(x) = e^{-x}$, find: i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$

21. Differentiate the following.

e $y = \cot\left(\frac{\pi}{4}-x\right)$ f $y = \sec(2x-\pi)$

22. Differentiate the following.

g $x^4 \operatorname{cosec}(4x)$ h $\tan 2x \cot x$ i $\sqrt{\sec x + \cos x}$

23. Differentiate the following.

a $e^{\sec x}$ b $\sec(e^x)$ c $e^x \sec x$

d $\cot(\ln x)$ e $\ln(\cot 5x)$ f $\cot x \ln x$

g $\operatorname{cosec}(\sin x)$ h $\sin(\operatorname{cosec} x)$ i $\sin x \operatorname{cosec} x$

Exercise 6.2.4

1. Differentiate with respect to *x*, each of the following.

g $\arccos\left(\frac{x}{4}\right)$ h $\arcsin\left(\frac{x+1}{2}\right)$ i $\text{Tan}^{-1}(x-4)$

j $\arcsin\left(\frac{2-x}{2}\right)$ k $\arctan\left(\frac{2x}{3}\right)$ l $\arccos\left(\frac{2x-1}{3}\right)$

2. Differentiate with respect to *x*, each of the following.

i $e^{\arcsin x}$ j $\frac{2}{\arctan(2x)}$ k $\frac{2}{\sqrt{\arcsin(x)}}$

l $\frac{1}{[\arccos x]^2}$ m $\cos(\text{Sin}^{-1}(2x))$ n $\sin(\arccos(2x))$

o $\tan(\arccos x)$

3. Differentiate with respect to *x*, each of the following.

g $e^x \arctan e^x$ h $(4+x^2)\arctan\left(\frac{x}{2}\right)$ i $\sqrt{4-x^2}\,\text{Sin}^{-1}\left(\frac{x}{2}\right)$

7. Differentiate the following and find the implied domain for each of $f(x)$ and $f'(x)$.

d $f(x) = \text{Sin}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ e $f(x) = \arcsin(ax), a \in \mathbb{R}$ f $f(x) = \arcsin(2x\sqrt{1-x^2})$

g $f(x) = \text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ h $f(x) = \text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$

8. Differentiate the following.

a $\arctan(x^n) + [\arctan(x)]^n$ b $\arcsin x + \arcsin\sqrt{1-x^2}$ c $x\sqrt{1-x^2} + \arcsin x$

d $\text{Tan}^{-1}\sqrt{\frac{x-b}{a-x}}, b < x < a$ e $\arctan[x - \sqrt{1+x^2}]$ f $\text{Cot}^{-1}x$

9. Given that $f : [-1, 1] \to \mathbb{R}$, where $f(x) = \arcsin(x)$ and $g(x) = \frac{1-x}{1+x}, x \in A$.

Find the largest set A such that $(f \circ g)(x)$ exists.

Exercise 6.2.5

10. Differentiate the following.

g $\sin(2^x)$ **h** $2^{\sin x}$ **i** $7^{\frac{1}{x}-2x}$

11. Differentiate the following.

g $\log_3(x^3-3)$ **h** $\log_2(\sqrt{2-x})$ **i** $\log_{10}\left(\cos\left(\frac{x}{2}-2\right)\right)$

12. Find the derivative of:

a x^x **b** $x^{\sin x}$ **c** $x^{\left(\frac{1}{x}\right)}$ **d** $x^{\ln x}$

Hint: Let $y = f(x)$ and then take log base e of both sides.

Exercise 6.2.6

1. Find the second derivative of the following functions.

m $f(x) = x^3 \sin x$ n $y = x \ln x$ o $f(x) = \frac{x^2 - 1}{2x + 3}$ p $y = x^3 e^{2x}$

q $f(x) = \frac{\cos(4x)}{e^x}$ r $y = \sin(x^2)$ s $f(x) = \frac{x}{1 - 4x^3}$ t $y = \frac{x^2 - 4}{x - 3}$

7. Find the nth derivative of:

a e^{ax} b $y = \frac{1}{2x + 1}$ c $\sin(ax + b)$

8. a Find $f''(2)$ if $f(x) = x^2 - \sqrt{x}$. b Find $f''(1)$ if $f(x) = x^2 \mathrm{Tan}^{-1}(x)$.

9. Find the rate of change of the gradient of the function $g(x) = \frac{x^2 - 1}{x^2 + 1}$ where $x = 1$.

10. Find the values of x where the rate of change of the gradient of the curve $y = x \sin x$ for $0 \le x \le 2\pi$ is positive.

Exercise 6.2.1

1
a $5x^4$ **b** $9x^8$ **c** $25x^{24}$ **d** $27x^2$
e $-28x^6$ **f** $2x^7$ **g** $2x$ **h** $20x^3 + 2$
i $-15x^4 + 18x^2 - 1$ **j** $-\frac{4}{3}x^3 + 10$
k $9x^2 - 12x$ **l** $3 + \frac{2}{5}x + 4x^3$

2
a $-\frac{3}{x^4}$ **b** $\frac{3}{2}\sqrt{x}$ **c** $\frac{5}{2}\sqrt{x^3}$ **d** $\frac{1}{3\sqrt[3]{x^2}}$
e $\frac{2}{\sqrt{x}}$ **f** $9\sqrt{x}$ **g** $\frac{1}{\sqrt{x}} + \frac{3}{x^2}$ **h** $\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x^3}}$
i $\frac{10}{3\sqrt[3]{x}} - 9$ **j** $5 - \frac{1}{2\sqrt{x}} - \frac{8}{5x^3}$ **k** $\frac{4}{\sqrt{x}} - \frac{15}{x^6} + \frac{1}{2}$ **l** $-\frac{1}{2\sqrt{x^3}} - \frac{1}{\sqrt{x}} + x^2$

3
a $\frac{3}{2}\sqrt{x} + \frac{1}{\sqrt{x}}$ **b** $4x^3 + 3x^2 - 1$ **c** $3x^2 + 1$ **d** $\frac{1}{x^2}$
e $\frac{1}{\sqrt{x^3}}$ **f** $\frac{1}{2} - \frac{1}{4\sqrt{x^3}}$ **g** -7 **h** $2x - \frac{8}{x^3}$
i $2x - \frac{2}{x^2} - \frac{4}{x^5}$ **j** $\frac{1}{2}\sqrt{\frac{3}{x}} + \frac{1}{6\sqrt{x^3}}$ **k** $2x - \frac{12}{5}\sqrt[5]{x} + \frac{2}{5\sqrt[5]{x^3}}$
l $-\frac{3}{2\sqrt{x}}\left(\frac{1}{x} + 1\right)\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)^2$

Exercise 6.2.2

1
a $48t^3 - \frac{1}{2\sqrt{t}}$ **b** $2n - \frac{2}{n^2} - \frac{4}{n^5}$ **c** $\frac{3}{2}\sqrt{r} + \frac{5}{6\sqrt[6]{r}} - \frac{1}{\sqrt{r}}$
d $2\theta - \frac{9}{2}\sqrt{\theta} + 3 - \frac{1}{2\sqrt{\theta}}$ **e** $40 - 3L^2$ **f** $-\frac{100}{v^3} - 1$
g $6l^2 + 5$ **h** $2\pi + 8h$

2
a $\frac{8}{3t^3}$ **b** $2\pi r - \frac{20}{r^2}$ **c** $\frac{5}{2}s^{3/2} + \frac{3}{s^2}$
d $-\frac{6}{t^4} + \frac{2}{t^3} - \frac{1}{t^2}$ **e** $-\frac{4}{b^2} + \frac{1}{2b^{3/2}}$ **f** $3m^2 - 4m - 4$

Exercise 6.2.3

1 **a** $3x^2 - 5x^4 + 2x + 2$ **b** $6x^5 + 10x^4 + 4x^3 - 3x^2 - 2x$

c $-\frac{4}{x^5}$ **d** $6x^5 + 8x^3 + 2x$

2 **a** $-\frac{2}{(x-1)^2}$ **b** $\frac{1}{(x+1)^2}$ **c** $\frac{1-x^2-2x}{(x^2+1)^2}$

d $\frac{-(x^4+3x^2+2x)}{(x^3-1)^2}$ **e** $\frac{2x^2+2x}{(2x+1)^2}$ **f** $\frac{1}{(1-2x)^2}$

3 **a** $(\sin x + \cos x)e^x$ **b** $\ln x + 1$ **c** $e^x(2x^3 + 6x^2 + 4x + 4)$

d $4x^3\cos x - x^4\sin x$ **e** $-\sin^2 x + \cos^2 x$ **f** $2x\tan x + (1 + x^2)\sec^2 x$

g $\frac{4}{x^3}(x\cos x - 2\sin x)$ **h** $e^x(x\cos x + x\sin x + \sin x)$

i $(\ln x + 1 + x\ln x)e^x$

4 **a** $\frac{\sin x - x\cos x}{\sin^2 x}$ **b** $\frac{-[\sin x(x+1) + \cos x]}{(x+1)^2}$ **c** $\frac{e^x}{(e^x+1)^2}$

d $\frac{2x\cos x - \sin x}{2x\sqrt{x}}$ **e** $\frac{\ln x - 1}{(\ln x)^2}$ **f** $\frac{(x+1) - x\ln x}{x(x+1)^2}$

g $\frac{xe^x + 1}{(x+1)^2}$ **h** $\frac{-2}{(\sin x - \cos x)^2}$ **i** $\frac{x^2 - x + 2x\ln x}{(x + \ln x)^2}$

5 **a** $-5e^{-5x} + 1$ **b** $4\cos 4x + 3\sin 6x$ **c** $-\frac{1}{3}e^{-\frac{1}{3}x} - \frac{1}{x} + 18x$

d $25\cos 5x + 6e^{2x}$ **e** $4\sec^2 4x + 2e^{2x}$ **f** $-4\sin(4x) + 3e^{-3x}$

g $\frac{4}{4x+1} - 1$ **h** 0 **i** $\frac{1}{2}\cos\left(\frac{x}{2}\right) - 2\sin 2x$

j $7\cos(7x - 2)$ **k** $\frac{1}{2\sqrt{x}} - \frac{1}{x}$ **l** $\frac{1}{x} + 6\sin 6x$

6 **a** $2x\cos x^2 + 2\sin x\cos x$ **b** $2\sec^2 2\theta - \frac{\cos\theta}{\sin^2\theta}$ **c** $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$

d $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$ **e** $-3\sin\theta \cdot \cos^2\theta$ **f** $e^x\cos(e^x)$

g $\frac{1}{x}\sec^2(\log_e x)$ **h** $\frac{-\sin 2x}{\sqrt{\cos 2x}}$ **i** $-\cos\theta \cdot \sin(\sin\theta)$

j $4\sin\theta \cdot \sec^2\theta$ **k** $-5\cos 5x \cdot \csc^2(5x)$ **l** $-6\csc^2(2x)$

7 **a** $2e^{2x+1}$ **b** $-6e^{4-3x}$ **c** $-12xe^{4-3x^2}$

d $\frac{1}{2}\sqrt{e^x}$ **e** $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **f** e^{2x+4}

g $2xe^{2x^2+4}$ **h** $-\frac{6}{e^{3x+1}}$ **i** $(6x-6)e^{3x^2-6x+1}$

j $\cos(\theta)e^{\sin\theta}$ **k** $2\sin(2\theta)e^{-\cos 2\theta}$ **l** $2x$

m $\frac{2e^{-x}}{(e^{-x}+1)^2}$ **n** $3(e^x+e^{-x})(e^x-e^{-x})^2$ **o** e^{x+2}

p $(-2x+9)e^{-x^2+9x-2}$

8 **a** $\frac{2x}{x^2+1}$ **b** $\frac{\cos\theta+1}{\sin\theta+\theta}$ **c** $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ **d** $-\frac{1}{x+1}$

e $\frac{3}{x}(\ln x)^2$ **f** $\frac{1}{2x\sqrt{\ln x}}$ **g** $\frac{1}{2(x-1)}$ **h** $\frac{-3x^2}{1-x^3}$

i $-\frac{1}{2(x+2)}$ **j** $\frac{-2\sin x\cos x}{\cos^2 x+1}$ **k** $\frac{1}{x}+\cot x$ **l** $\frac{1}{x}+\tan x$

9 **a** $\ln(x^3+2)+\frac{3x^3}{x^3+2}$ **b** $\frac{\sin^2 x}{2\sqrt{x}}+2\sqrt{x}\sin x\cos x$ **c** $-\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\cdot\cos\sqrt{\theta}$

d $(3x^2-4x^4)e^{-2x^2+3}$ **e** $-(\ln x+1)\sin(x\ln x)$ **f** $\frac{1}{x\ln x}$

g $\frac{(2x-4)\cdot\sin(x^2)-2x\cdot\cos(x^2)(x^2-4x)}{(\sin x^2)^2}$ **h** $\frac{10(\ln(10x+1)-1)}{[\ln(10x+1)]^2}$

i $(\cos 2x-2\sin 2x)e^{x-1}$ **j** $2x\ln(\sin 4x)+4x^2\cot 4x$ **k** $(\cos\sqrt{x}-\sin\sqrt{x})\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

l $-(2\sin x+2x\cos x)\cdot\sin(2x\sin x)$ **m** $\frac{e^{5x+2}(9-20x)}{(1-4x)^2}$ **n** $\frac{\cos^2\theta+\sin^2\theta\ln(\sin\theta)}{\sin\theta\cos^2\theta}$

o	$\frac{x+2}{2(x+1)\sqrt{x+1}}$	**p**	$\frac{2x^2+2}{\sqrt{x^2+2}}$	**q**	$\frac{10x^3+9x^2+4x+3}{3(x+1)^{2/3}}$
r	$\frac{3x^2(3x^3+1)}{2\sqrt{x^3+1}}$	**s**	$\frac{2}{x^2+1}-\frac{1}{x^2}\ln(x^2+1)$	**t**	$\frac{2}{x(x+2)}$
u	$\frac{2-x}{2x^2\sqrt{x-1}}$	**v**	$\frac{-x^2+x-9}{\sqrt{x^2+9}}\cdot e^{-x}$	**w**	$\frac{7x^3-12x^2-8}{2\sqrt{2-x}}$
x	$nx^{n-1}\ln(x^n-1)+\frac{nx^{2n-1}}{x^n-1}$				

10 $x = 1$

11 0

12 0

13 1

14 $-2e$

15 **a** $\cos^2x-\sin^2x$ **b** $\frac{\pi}{180}\cos x°$ **c** $-\frac{\pi}{180}\sin x°$

16 **b i** $2x\sin x\cos x + x^2\cos^2x - x^2\sin^2x$ **ii** $e^{-x^3}(2\cos 2x\ln\cos x - 3x^2\sin 2x\ln\cos x - \sin 2x\tan x)$

17 **a i** $-\frac{3}{x}(\ln x)^2$ **ii** $-\frac{3x^2}{1-x^3}$ **b i** $-2e^{-2x}\cdot\cos(e^{-2x})$ **ii** $-2x\cos x^2\cdot e^{-\sin x^2}$

18 $-\frac{1}{5}k$

19 $x = a, b, \frac{mb+na}{m+n}$

20 $\{\theta : n\tan\theta^m\cdot\tan\theta^n = m\theta^{m-n}\}$

21 **a** $-4\csc(4x)$ **b** $2\sec(2x)\tan(2x)$ **c** $3\cot(3x)\csc(3x)$ **d** $-3\sin(3x)$
e $\csc^2\left(\frac{\pi}{4}-x\right)$ **f** $-2\sec(2x)\tan(2x)$

22 **a** $2x\sec(x^2)\tan(x^2)$ **b** $\sec^2x$ **c** $\tan x$
d $-3\cot^2x\csc^2x$ **e** $x\cos x+\sin x$ **f** $-2\cot x\csc^2x$
g $4x^3\csc(4x)-4x^4\cot(4x)\csc(4x)$ **h** $2\cot x\sec^2(2x)-\csc^2x\tan(2x)$
i $\frac{\sec x\tan x-\sin x}{2\sqrt{\cos x+\sec x}}$

23 **a** $e^{\sec x}\sec x\tan x$ **b** $e^x\sec(e^x)\tan(e^x)$ **c** $e^x\sec(x)+e^x\sec(x)\tan(x)$
d $\frac{-\csc^2(\log x)}{x}$ **e** $-5\csc(5x)\sec(5x)$ **f** $\frac{\cot(x)}{x}-\csc^2(x)\log x$
g $-\cos x\cot(\sin x)\csc(\sin x)$ **h** $-\cos(\csc x)\cot x\csc x$ **i** 0

Exercise 6.2.4

1 **a** $\frac{2}{4x^2+1}$ **b** $\frac{1}{\sqrt{9-x^2}}$ **c** $\frac{-2}{\sqrt{1-4x^2}}$ **d** $\frac{4}{\sqrt{1-16x^2}}$

e $\frac{2}{x^2+4}$ **f** $\frac{1}{\sqrt{2x-x^2}}$ **g** $\frac{-1}{\sqrt{16-x^2}}$ **h** $\frac{1}{\sqrt{4-(x+1)^2}}$

i $\frac{1}{(4-x)^2+1}$ **j** $\frac{-1}{\sqrt{4x-x^2}}$ **k** $\frac{6}{4x^2+9}$ **l** $\frac{-1}{\sqrt{-x^2+x+2}}$

2 **a** $\frac{2x}{x^4+1}$ **b** $\frac{1}{2\sqrt{x}-x^2}$ **c** $\frac{1}{2\sqrt{x^3}-x^2}$ **d** $\frac{-\sin x}{\sqrt{1-\cos^2 x}} = \begin{cases}-1 \text{ if } \sin x > 0\\ 1 \text{ if } \sin x < 0\end{cases}$

e $\frac{1}{2x\sqrt{x-1}}$ **f** $\frac{1}{\sqrt{1-x^2}\,\text{Sin}^{-1}x}$ **g** $\frac{e^x}{1+e^{2x}}$ **h** $\frac{1}{\sqrt{e^{2x}-1}}$

i $\frac{e^{\arcsin x}}{\sqrt{1-x^2}}$ **j** $\frac{-4}{(4x^2+1)[\tan^{-1}(2x)]^2}$ **k** $\frac{-1}{\sqrt{1-x^2}(\sin^{-1}(x))^{3/2}}$

l $\frac{2}{\sqrt{1-x^2}(\cos^{-1}(x))^3}$ **m** $\frac{-4x}{\sqrt{1-4x^2}}$ **n** $\frac{-4x}{\sqrt{1-4x^2}}$

o $\frac{-1}{x^2\sqrt{1-x^2}}$

3 **a** $\text{Tan}^{-1}x + \frac{x}{1+x^2}$ **b** $\frac{x-\sqrt{1-x^2}\sin^{-1}x}{x^2\sqrt{1-x^2}}$ **c** $\frac{x+\sqrt{1-x^2}\cos^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$

d $\frac{-2x^2\tan^{-1}x + x - 2\tan^{-1}x}{x^3(x^2+1)}$ **e** $\frac{2x^2\log x + \sqrt{1-x^4}\sin^{-1}(x^2)}{x\sqrt{1-x^4}}$

f $\frac{-\sqrt{1-x}\cos^{-1}\sqrt{x}-\sqrt{x}}{2x^{3/2}\sqrt{1-x}}$ **g** $e^x\tan^{-1}(e^x) + \frac{e^{2x}}{1+e^{2x}}$ **h** $2x\tan^{-1}\left(\frac{x}{2}\right)+2$

i $1 - \frac{x}{\sqrt{4-x^2}}\text{Sin}^{-1}\left(\frac{x}{2}\right)$

4 $0, k = \frac{\pi}{2}$

6 **b** $k = \frac{\pi}{2}$

7 **a** $f'(x) = \frac{-\pi}{x\sqrt{x^2-\pi^2}}, x > \pi$ and $\frac{\pi}{x\sqrt{x^2-\pi^2}}, x < -\pi$; $\text{dom}(f) =]-\infty, -\pi[\cup]\pi, \infty[$

b $f'(x) = \frac{1}{x\sqrt{2x-1}}, x > \frac{1}{2}$; $\text{dom}(f') = \left]\frac{1}{2}, \infty\right[$, $\text{dom}(f) = \left[\frac{1}{2}, \infty\right[$

c $f'(x) = \frac{1}{\sqrt{1-x^2}}\text{Cos}^{-1}\left(\frac{x}{2}\right) - \frac{1}{\sqrt{4-x^2}}\text{Sin}^{-1}(x), -1 < x < 1$; $\text{dom}(f) = [-1, 1]$

d $f'(x) = -\frac{2}{x^2+1}, x > 0$ and $\frac{2}{x^2+1}, x < 0$; $\text{dom}(f) =]-\infty, \infty[$

e $f'(x) = \frac{a}{\sqrt{1-a^2x^2}}, |x| < \frac{1}{a}$; $\text{dom}(f) = \left[-\frac{1}{a}, \frac{1}{a}\right]$

f $f'(x) = \frac{2}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f'(x) = \frac{-2}{\sqrt{1-x^2}}, -1 < x < -\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}} < x < 1$; $\text{dom}(f) = [-1, 1]$

g $f'(x) = \frac{2}{x^2+1}, x > 0$ and $\frac{-2}{x^2+1}, x < 0$; $\text{dom}(f) =]-\infty, \infty[$

h $f'(x) = \frac{2}{x^2+1}, |x| < 1$ and $\frac{-2}{x^2+1}, |x| > 1$; $\text{dom}(f) =]-\infty, \infty[$

8 a $\frac{nx^{n-1}}{1+x^{2n}} + \frac{n}{1+x^2}(\arctan x)^{n-1}$ b 0 c $2\sqrt{1-x^2}$ d $\frac{1}{2\sqrt{(a-x)(x-b)}}$

e $\frac{1}{2(1+x^2)}$ f $-\frac{1}{x^2+1}$

9 a $x \geq 0$ b $\frac{-1}{(x+1)\sqrt{x}}, x > 0$

Exercise 6.2.5

1 a $(\ln 4)4^x$ b $(\ln 3)3^x$ c $(\ln 8)8^x$ d $3(\ln 5)5^x$

e $7(\ln 6)6^x$ f $2(\ln 10)10^x$ g $(\ln 6)6^{x-2}$ h $3(\ln 2)2^{3x+1}$

i $-5(\ln 7)7^{3-x}$

2 a $x(\ln 3)3^x + 3^x$ b $4\cos(2x)2^x + 2\ln(2)\sin(2x)2^x$ c $\ln(5)5^x e^{-x} - 5^x e^{-x}$

d $2 \times 8^{-x} - \ln(8)8^{-x}x^2$ e $\frac{(1+4^x)-(x+2)\ln(4)4^x}{(1+4^x)^2}$ f $-\sin x 5^x + \ln(5)5^x \cos x$

3 a $\frac{1}{(\ln 5)x}$ b $\frac{1}{(\ln 10)x}$ c $\frac{1}{(\ln 4)x}$ d $\frac{1}{(\ln 9)(x+1)}$

e $\frac{2x}{(\ln 2)(x^2+1)}$ f $\frac{1}{2(\ln 5)(x-5)}$ g $\log_2 x + \frac{1}{\ln 2}$ h $(\ln 3)3^x \log_3 x + \frac{3^x}{(\ln 3)x}$

i $(\ln a)a^x\log_a x+\dfrac{a^x}{(\ln a)x}$ **j** $\dfrac{(\ln a)^2xa^x\log_a x-a^x}{(\ln a)x(\log_a x)^2}$ **k** $\dfrac{(\ln(10))\log_{10}(x+1)-1}{\ln(10)(\log_{10}(x+1))^2}$

l $\dfrac{(\ln(2))2\log_2 x-2}{\ln(2)(\log_2 x)^2}$

4 $\dfrac{1}{\ln 2}$

5 $0, -\dfrac{2}{\ln 2}$

6 $\dfrac{1-\ln 3}{3}$

7 $\pi 2^{-\pi}\sqrt{3}+\dfrac{\sqrt{3}\pi\ln\pi}{2}$

8 1.25

9 **a** $20+10\ln 10$ **b** $(\ln 4)\cos(1)$ **c** $\dfrac{1}{2}$ **d** $10-\dfrac{10}{\ln 10}$

10 **a** $4\times 5^{4x+1}\ln 5$ **b** $3^{x-x^3}(1-3x^2)\ln 3$ **c** $2(10^{2x-3})\ln 10$

d $9^{\sqrt{x}-x}\left(\dfrac{1}{2\sqrt{x}}-1\right)\ln 9$ **e** $-2^{\cos(2x)+1}\ln 2\sin 2x$ **f** $\dfrac{-4^{\sqrt{\cos 2x}}\ln 4\sin 2x}{\sqrt{\cos 2x}}$

g $2^x\cos 2^x\ln 2$ **h** $2^{\sin x}\cos x\ln 2$ **i** $-7^{\left(\frac{1}{x}-2x\right)}(2+x^{-2})\ln 7$

11 **a** $\dfrac{2\cot 2x}{\ln 2}$ **b** $\dfrac{x}{(x^2-1)\ln 5}$ **c** $\dfrac{1}{2(\sqrt{x}-10)\sqrt{x}\ln 10}$

d $\dfrac{-4\sec^2 2x}{\ln 2(4-2\tan 2x)}$ **e** $\dfrac{1}{2\sqrt{x-x^2}\sin^{-1}\sqrt{x}\ln 2}$ **f** $\dfrac{-1}{((1-x^2)+1)\tan^{-1}(1-x)\ln 3}$

g $\dfrac{3x^2}{(x^3-3)\ln 3}$ **h** $\dfrac{-1}{2(2-x)\ln 2}$ **i** $-\dfrac{1}{2\ln 10}\tan\left(\dfrac{x}{2}-2\right)$

12 **a** $x^x(\ln x+1)$ **b** $x^{\sin x}\left(\cos x\ln x+\dfrac{\sin x}{x}\right)$ **c** $(1-\ln x)x^{\frac{1}{x}-2}$ **d** $2\ln(x)x^{\ln x-1}$

Exercise 6.2.6

1 **a** $20x^3$ **b** $48(1+2x)^2$ **c** $\frac{2}{x^3}$ **d** $\frac{2}{(1+x)^3}$

e 2 **f** $\frac{6}{(x-2)^3}$ **g** $\frac{42}{x^8}$ **h** $24(1-2x)$

i $-\frac{1}{x^2}$ **j** $\frac{2(x^2+1)}{(1-x^2)^2}$ **k** $-16\sin 4\theta$ **l** $2\cos x - x\sin x$

m $6x^2\cos x + 6x\sin x - x^3\sin x$ **n** $\frac{1}{x}$ **o** $\frac{10}{(2x+3)^3}$

p $6x\mathrm{e}^{2x} + 12x^2e^{2x} + 4x^3e^{2x}$ **q** $\frac{8\sin 4x - 15\cos 4x}{e^x}$

r $2\cos x^2 - 4x^2\sin x^2$ **s** $\frac{-48(x^2+2x^5)}{(4x^3-1)^3}$ **t** $\frac{10}{(x-3)^3}$

2 **a** $\frac{-2x}{(x^2+1)^2}$ **b** $\frac{x}{(1-x^2)^{3/2}}$ **c** $\frac{-x}{(1-x^2)^{3/2}}$ **d** $\frac{2}{(x^2+1)^2}$

e $\frac{2x-1}{4(x-x^2)^{3/2}}$ **f** $\frac{2x-3x^2}{4\sqrt{(x^3-x^2)^3}}$

3 $\frac{6\ln x - 5}{x^4}, \frac{n^2\ln x + n\ln x - 2n - 1}{x^{n+2}}$

4 $f'(x) = -\frac{1}{(x+1)^2}, f''(x) = \frac{2}{(x+1)^3}, f^{(iii)}(x) = -\frac{6}{(x+1)^4}, f^{(iv)}(x) = \frac{24}{(x+1)^5}$

$f^{(v)}(x) = -\frac{120}{(x+1)^6}, \ldots\ldots, f^{(n)}(x) = (-1)^n\frac{n!}{(x+1)^{n+1}}$

5 $f(x) = \left(\frac{x+1}{x-1}\right)^n \Rightarrow f''(x) = \frac{4n(n+x)}{(x^2-1)^2}\left(\frac{x+1}{x-1}\right)^n$

7 **a** $a^n e^{ax}$ **b** $\frac{(-1)^n 2^n n!}{(2x+1)^{n+1}}$

c

$n = 2k: \quad y^{(n)}(x) = (-1)^k a^{2k}\sin(ax+b),\ k = 1, 2, \ldots$

$n = 2k-1: \quad y^{(n)}(x) = (-1)^{k+1}a^{2k-1}\cos(ax+b),\ k = 1, 2, \ldots$

8 **a** $2 + \frac{1}{8\sqrt{2}}$ **b** $\frac{3+\pi}{2}$

9 -1 [0,1.0768[∪]3.6436,2p]

Exercise 6.2.7

1 **a** $-2x$ **b** $-\frac{x}{y}$ **c** $\frac{1}{x^3y}$ **d** $-\frac{y}{x+1}$

e $-\frac{ye^x}{1+e^x}$ **f** $\frac{\sin x - y}{x}$ **g** $-x$ **h** $\frac{1-3x^4y}{x^5}$

i $-\frac{y\cos x + 2}{\sin x}$ **j** -1 **k** $\frac{4x^3}{3y^2+1}$ **l** $\sqrt{x+y}-1$

2 (1,5), 0

4 $\left(\frac{3-2\sqrt{10.6}}{2}, \frac{80+4\sqrt{265}}{40}\right), \left(\frac{3+2\sqrt{10.6}}{2}, \frac{80-4\sqrt{265}}{40}\right)$

5 **a** $y = \frac{x \pm \sqrt{5x^2-80}}{2}$ **c** $\frac{dy}{dx} = \frac{2x+y}{2y-x}$ **d** $\frac{5x \pm \sqrt{5x^2-80}}{2\sqrt{5x^2-80}}$

e Hyperbola

6 **a** Dom = Ran = [−2,2] **b** $-\frac{x^3}{y^3}$ **c** $-\frac{x^3}{(\sqrt[4]{16-x^4})^3}$ **d** small

e Dom = Ran = $[-k,k]$ **f** $\frac{dy}{dx} = -\frac{x^{2n-1}}{y^{2n-1}}$

7 **a** $\frac{-\nu}{p\gamma}$ **b** $\frac{n(m-1)x^{m-2}}{m(n+1)y^n}$

8 **a** $\frac{1}{11}$ **b** -1

9 **a** $\frac{y}{xy-x}$ **b** $\frac{(1+y^2)(\tan^{-1}y - 1)}{1-x+y^2}$

10 **a** undefined **b** At (0.8042, 0.5), grad = 1.32; at (0.0646, 0.5), grad = 3.74

Exercise 6.3.1

6. Find the equation of the tangent and the normal to the curve $x \mapsto x + \frac{1}{x}, x \neq 0$ at the point (1, 2).

   Find the coordinates of the points where the tangent and the normal cross the x- and y-axes, and hence determine the area enclosed by the x-axis, the y-axis, the tangent and the normal.

7. Find the equation of the normal to the curve $y = \sqrt{25 - x^2}$ at the point (4, 3).

8. Show that every normal to the curve $y = \sqrt{a^2 - x^2}$ will always pass through the point (0, 0).

9. Find the equation of the tangent to the curve $y = x^2 - 2x$ that is parallel to the line with equation $y = 4x + 2$.

10. Find the equation of the tangent to the curve $x \mapsto \log_e(x^2 + 4)$ at the point where the curve crosses the y-axis.

11. Find the equation of the tangent and the normal to the curve $x \mapsto x\tan(x)$ where $x = \frac{\pi}{4}$.

12. The straight line $y = -x + 4$ cuts the parabola with equation $y = 16 - x^2$ at the points A and B.

    a Find the coordinates of A and B.

    b Find the equation of the tangents at A and B, and hence determine where the two tangents meet.

13. For the curve defined by:

$$x \mapsto \frac{x}{x^2 + 1}$$

   find the equation of the normal at the origin, and the equations of the tangents that are parallel to the x-axis. Find also the points where the tangents and the normal intersect.

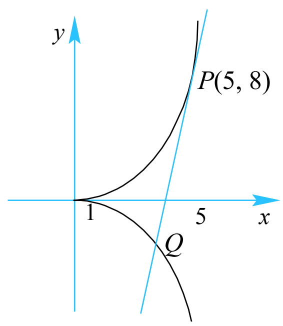
14. The figure shows the curve whose equation is given by:

$$y^2 = (x - 1)^3.$$

   The tangent drawn at the point $P(5, 8)$ meets the curve again at the point Q.

    a Find the equation of the tangent at the point P.

    b Find the coordinates of Q.



15. The line L and the curve C are defined as follows:

$L: y = 4x - 2$ and $C: y = mx^3 + nx^2 - 1$

The line L is a tangent to the curve C at $x = 1$.

a Using the fact that L and C meet at $x = 1$, show that $m + n = 3$.

b Given that L is a tangent to C at $x = 1$, show that $3m + 2n = 4$.

c Hence, solve for m and n.

16. For each of the following curves, find the equation of the normal at the points indicated:

a $x^2 + 2y^2 = 9$ at the point (1, 2).

b $2xy - \sqrt{x^2 + y^2} = 19$ at the point (3, 4).

c $4(x + y) + 3xy = 0$ at the point (–1, 4).

d $x = \tan\left(\frac{x}{y}\right)$ at the point $\left(1, \frac{4}{\pi}\right)$

e $x^2 + 3xy^2 - 4 = 0$ at the point (1, 1).

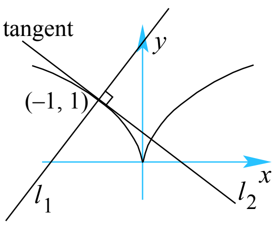
17. For the curve $x^2 + y^2 - xy = 3$, find:

a the equation of the normal at (2, 1).

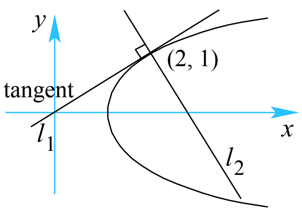
b the equations of the tangents to the curve that are parallel to the x-axis.

18 Find the equation of the lines l_1 and l_2 in each of the following situations.

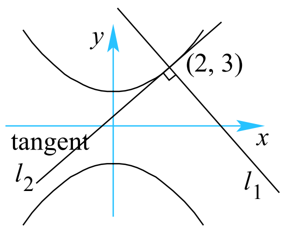
a $x^2 - y^3 = 0$



b $x - y^2 = 1$



c $4y^2 - x^2 = 32$



19, Find the point of intersection of the tangents to the curve $y^2 - 3xy + x^3 = 3$ at the points where $x = -1$.

Exercise 6.3.2

2. Find the coordinates and nature of the stationary points for the following:

j $y = x\sqrt{x} - x, x \geq 0$ k $g(x) = x + \frac{4}{x}, x \neq 0$ l $f(x) = x^2 + \frac{1}{x^2}, x \neq 0$

3. Sketch the following functions:

e $f(x) = \frac{1}{3}x^3 - x^2 + 4$ f $y = 4x^3 - x^4$ g $y = x^3 - 8$

h $y = x^4 - 16$ i $y = x - 4x\sqrt{x}, x \geq 0$ j $f(x) = x - 2\sqrt{x}, x \geq 0$

8. A function f is defined by $f: x \mapsto e^x \cos x$, where $0 \leq x \leq 2\pi$.

a Find: i $f'(x)$ ii $f''(x)$

b Find the values of x for which:

i $f'(x) = 0$ ii $f''(x) = 0$.

c Using parts a and b, find the points of inflection and stationary points for f.

d Hence, sketch the graph of f.

9. A function f is defined by $f: x \mapsto xe^{-x}$, where $x > 0$.

a Find: i $f'(x)$ ii $f''(x)$

b Find the values of x for which:

i $f'(x) = 0$ ii $f''(x) = 0$

c Using parts a and b, find the points of inflection and stationary points for f.

d Hence, sketch the graph of f.

10. a Find the maximum value of the function $y = 6x - x^2, 4 \leq x \leq 7$.

b Find the minimum value of the function $y = 6x - x^2, 2 \leq x \leq 6$.

c Find the maximum value of the function $y = 2x - x^3, -2 \leq x \leq 6$.

d Find the maximum value of the function $y = 36x - x^4, 2 \leq x \leq 3$.

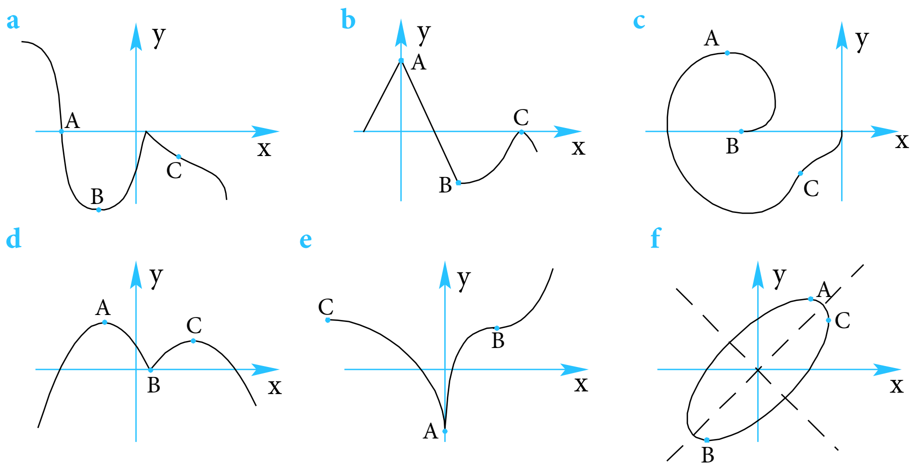
11. For the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 8, -6 \leq x \leq 6$, find:

a its minimum value. b its maximum value.

12. For each of the labelled points on the following graphs state:

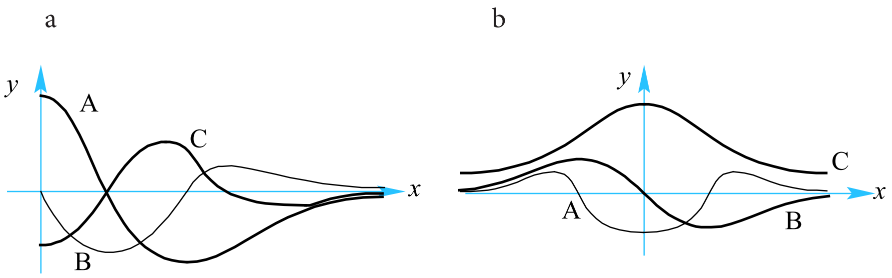
i whether the derivative exists at the point.

ii the nature of the curve at the point.

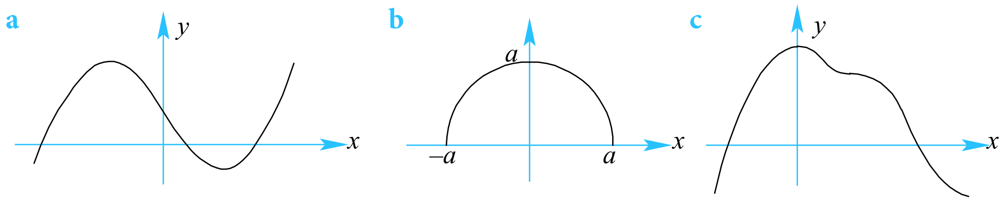


13. Identify which graph corresponds to:

i $f(x)$ ii $f'(x)$ iii $f''(x)$



14. For each of the functions, $f(x)$, sketch: i $f'(x)$ ii $f''(x)$



15. The curve with equation $y = ax^3 + bx^2 + cx + d$ has a local maximum where $x = -3$ and a local minimum where $x = -1$. If the curve passes through the points (0, 4) and (1, 20) sketch the curve for $x \in \mathbb{R}$.

16. The function $f(x) = ax^3 + bx^2 + cx + d$ has turning points at $\left(-1, -\frac{13}{3}\right)$ and (3, –15). Sketch the graph of the curve $y = f(x)$.

17. The function $f: x \mapsto \mathbb{R}$, where $f(x) = ax^5 + bx^3 + cx$ has stationary points at (–2, 64) (2, –64) and (0, 0). Find the values of a, b and c and hence sketch the graph of f.

18. Sketch the graph of the curve defined by the equation $y = x(10x - \ln x), x > 0$ identifying, where they exist, all stationary points and points of inflection.

19. Find m and n so that $f'(1)$ exists for the function $f(x) = \begin{cases} mx^2 + n & \text{if } x \le 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}$.

Exercise 6.3.3

17. A solid ball of radius 30 cm is dissolving uniformly in such a way that its radius is x cm, and is decreasing at a constant rate of 0.15 cm/s, t seconds after the process started.

 a Find an expression for the radius of the ball at any time t seconds.

 b Find the domain of x.

 c Find the rate of change of:

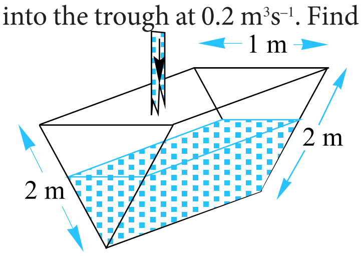
  i the volume of the ball 10 seconds after it started to dissolve.

  ii the surface area when the ball has a volume of 100π cm^3.

 d Sketch a graph of the volume of the ball at time t seconds.

18. A fisherman is standing on a jetty and is pulling in a boat by means of a rope passing over a pulley. The pulley is 3 m above the horizontal line where the rope is tied to the boat. At what rate is the boat approaching the jetty if the rope is being hauled at 1.2 m/s, when the rope measures 12 m?

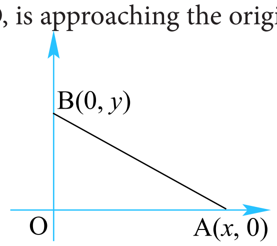
19. A trough, 4 m long, has a cross-section in the shape of an isosceles triangle. Water runs into the trough at 0.2 m^{3}s^{-1}. Find the rate at which the water level is rising after 10 seconds if the tank is initially empty.



20. A line, 12 m long, meets the x-axis at A and the y-axis at B. If point A, initially 5 m from O, is approaching the origin, O, at 2 m/s, find:

 a an expression for y in terms of the time, t seconds, since point A started to move.

 b the rate at which B is moving when A has travelled 2 m.



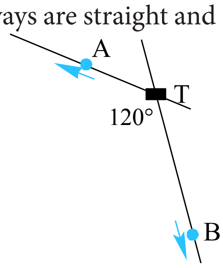
21^: The volume V cm^3 of water in a container at time t seconds, when the depth of water in the container is x cm is given by the relationship

$$V = \frac{1}{3}(x+3)^3 - 9\,,\ 0 \le x \le 5.$$

 a Find the rate at which the water level is increasing after 5 seconds if water flows into the container at 1.2 cm^{3}s^{-1}.

 b Find the rate of change of the area of the surface of the water after 5 seconds if water is still flowing into the container at 1.2 cm^{3}s^{-1}.

22. Two cars, A and B, leave their hometown, T, at the same time but on different freeways. The freeways are straight and at 120° to each other and the cars are travelling at 70 km/h and 80 km/hr respectively. Given that x km and y km are the distances travelled by the cars A and B respectively t hours after they leave T:



 a find an expression in terms of t for the distance travelled by car:

  i A ii B

 b find an expression in terms of t for the distance apart cars A and B are after t hours.

 c How fast are cars A and B moving apart after 5 hours?

 d After travelling for 5 hours, the driver of car B decides to head back to T. How fast are the cars moving apart 3 hours after car B turns back?

23. A girl approaches a tower 75 m high at 5 km/hr. At what rate is her distance from the top of the tower changing when she is 50 m from the foot of the tower?

24. Jenny is reeling in her kite, which is maintaining a steady height of 35 m above the reel. If the kite has a horizontal speed of 0.8 m/s towards Jenny, at what rate is the string being reeled in when the kite is 20 m horizontally from Jenny?

25. A kite 60 metres high, is being carried horizontally away by a wind gust at a rate of 4 m/s. How fast is the string being let out when the string is 100 m long?

26. Grain is being released from a chute at the rate of 0.1 cubic metres per minute and is forming a heap on a level horizontal floor in the form of a circular cone that maintains a constant semi-vertical angle of 30°. Find the rate at which the level of the grain is increasing 5 minutes after the chute is opened.

27. A radar tracking station is located at ground level vertically below the path of an approaching aircraft flying at 850 km/h and maintaining a constant height of 9,000 m. At what rate in degrees is the radar rotating while tracking the plane when the horizontal distance of the plane is 4 km from the station.

28. A weather balloon is released at ground level and 2,500 m from an observer on the ground. The balloon rises straight upwards at 5 m/s. If the observer is tracking the balloon from his fixed position, find the rate at which the observer's tracking device must rotate so that it can remain in-line with the balloon when the balloon is 400 m above ground level.

29. The radius of a uniform spherical balloon is increasing at 3% per second.

 a Find the percentage rate at which its volume is increasing.

 b Find the percentage rate at which its surface area is increasing.

30. A manufacturer has agreed to produce x thousand 10-packs of high quality recordable compact discs and have them available for consumers every week with a wholesale price of $\$k$ per 10-pack. The relationship between x and k has been modelled by the equation $x^2 - 2.5kx + k^2 = 4.8$

 At what rate is the supply of the recordable compact discs changing when the price per 10-pack is set at \$9.50, 4420 of the 10-pack discs are being supplied and the wholesale price per 10-pack is increasing at 12 cents per 10-pack per week?

31. It has been estimated that the number of housing starts, N millions, per year over the next 5 years will be given by:

$$N(r) = \frac{8}{1 + 0.03r^2},$$

 where r% is the mortgage rate. The government believes that over the next t months, the mortgage rate will be given by:

$$r(t) = \frac{8.6t + 65}{t + 10}.$$

 Find the rate at which the number of housing starts will be changing 2 years from when the model was proposed.

32. The volume of a right circular cone is kept constant while the radius of the base of the cone is decreasing at 2% per second. Find the percentage rate at which the height of the cone is changing.

33. The radius of a sector of fixed area is increasing at 0.5 m/s. Find the rate at which the angle in radians of the sector is changing when the ratio of the radius to the angle is 4.

Exercise 6.3.1

1 **a** $y = 7x - 10$ **b** $y = -4x + 4$ **c** $4y = x + 5$

d $16y = -x + 21$ **e** $4y = x + 1$ **f** $4y = x + 2$

g $y = 28x - 48$ **h** $y = 4$

2 **a** $7y = -x + 30$ **b** $4y = x - 1$ **c** $y = -4x + 14$

d $y = 16x - 79$ **e** $2y = 9 - 8x$ **f** $y = -4x + 9$

g $28y = -x + 226$ **h** $x = 2$

3 **a** $y = 2ex - e$ **b** $y = e$ **c** $y = \pi$ **d** $y = -x$

e $y = x$ **f** $ey = (2e-1)x - e^2 + 2e - 1$

g $y = ex$ **h** $y = 2x + 1$

4 **a** $2ey = -x + 2e^2 + 1$ **b** $x = 1$ **c** $x = \pi$ **d** $y = x - 2\pi$ **e** $y = -x + \pi$

f $(2e-1)y = -ex + 3e^2 - 4e + 1$ **g** $ey = -x$ **h** $2y = -x + 2$

5 A: $y = 28x - 44$, B: $y = -28x - 44$, Isosceles. $z \equiv (0, a^2 - 3a^4)$

6 2 sq. units, $y = 2$ $x = 1$

7 $4y = 3x$

8 $by = \sqrt{a^2 - b^2}x$

9 $y = 4x - 9$

10 $y = \log_e 4$

11 $8y = 4(\pi+2)x - \pi^2$; $4(\pi+2)y = -8x + 4\pi + \pi^2$

12 A: $y = -8x + 32$, B: $y = 6x + 25$, $\left(\frac{1}{2}, 28\right)$

13 $y = -x$, Tangents: $y = \frac{1}{2}, y = -\frac{1}{2}$ $\left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)$ tangent and normal meet at $(0.5, -0.5)$

14 **a** $y = 3x - 7$ **b** $Q \equiv (2, -1)$

15 $m = -2$, $n = 5$

16 **a** $y = 4x - 2$ **b** $37y = 26x + 70$

c $16y = x + 65$ **d** $y = \frac{4}{\pi} + \frac{\pi^2}{4(\pi-2)} - \frac{\pi^2}{4(\pi-2)}x$ **e** $5y = 6x - 1$

17 **a** $y = 1$ **b** At $(1, 2)$ $y = 2$; At $(-1, -2)$ $y = -2$

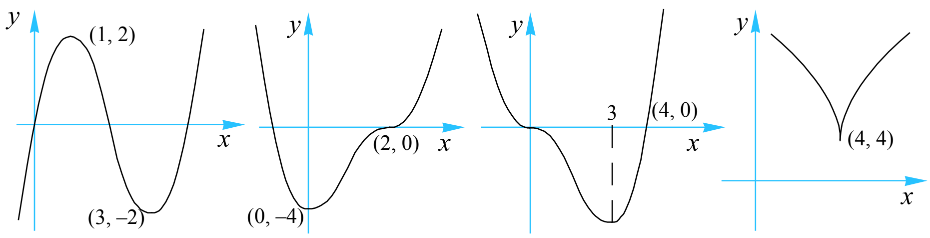
18 **a** $l_1: 3y = -2x + 1$, $l_2: 2y = 3x + 5$ **b** $l_1: 2y = x$, $l_2: y = -2x + 5$

c $l_1: 6y = x + 16$, $l_2: y = -6x + 15$

19 $\left(\frac{2}{3}, 1\right)$

Exercise 6.3.2

1 **a** **b** **c** **d**



2 **a** max at (1, 4) **b** min at $\left(-\frac{9}{2}, -\frac{81}{4}\right)$ **c** min at (3, –45) max (–3, 63)

d max at (0, 8), min at (4, -24) **e** max at (1, 8), min at (–3, –24)

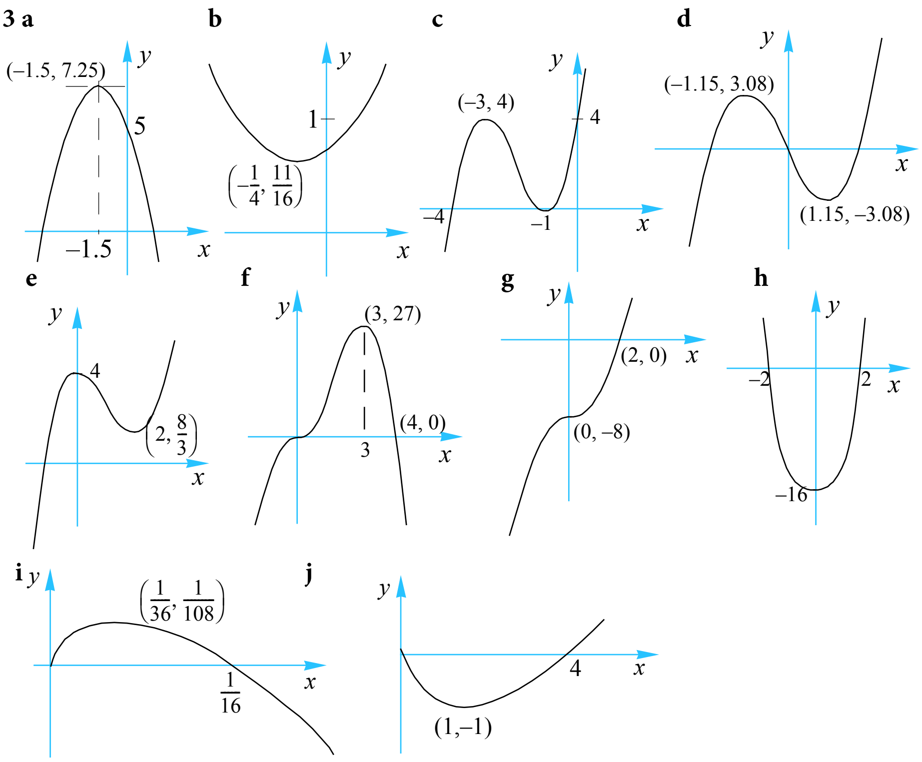
f min at $\left(\frac{1+\sqrt{13}}{3}, \frac{70-26\sqrt{13}}{27}\right)$, max at $\left(\frac{1-\sqrt{13}}{3}, \frac{70+26\sqrt{13}}{27}\right)$ **g** min at (1, –1)

h max at (0, 16), min at (2, 0), min at (–2, 0) **i** min at (1, 0) max at $\left(-\frac{1}{3}, \frac{32}{27}\right)$

j min at $\left(\frac{4}{9}, -\frac{4}{27}\right)$ **k** min at (2, 4), max at (–2, –4)

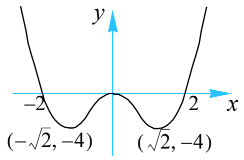
l min at (1, 2), min at (–1, 2)

3



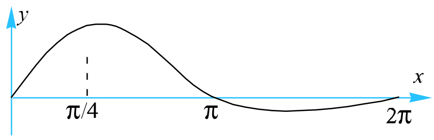
4 min at $(1, -3)$, max at $(-3, 29)$, non-stationary infl $(-1, 13)$

5



6 **a i** $(\cos x - \sin x)e^{-x}$ **ii** $-2\cos x.e^{-x}$ **b i** $\frac{\pi}{4}, \frac{5\pi}{4}$ **ii** $\frac{\pi}{2}, \frac{3\pi}{2}$

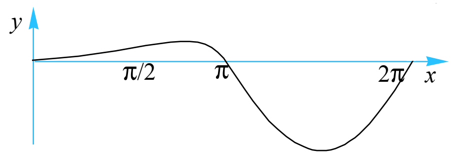
c Inf. $\left(\frac{\pi}{2}, e^{-\frac{\pi}{2}}\right)$ $\left(\frac{3\pi}{2}, -e^{-\frac{3\pi}{2}}\right)$ **d**



7 **a i** $e^x(\sin x + \cos x)$ **ii** $2e^x\cos x$ **b i** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **ii** $x = \frac{\pi}{2}, \frac{3\pi}{2}$

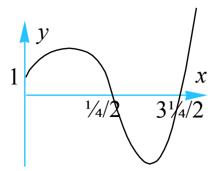
c St. pts. $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right), \left(\frac{7\pi}{4}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$ Infl. pts. $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right), \left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$

d



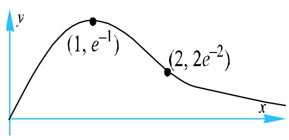
8 **a i** $e^x(\cos x - \sin x)$ **ii** $-2\sin x.e^x$ **b i** $\frac{\pi}{4}, \frac{5\pi}{4}$ **ii** $0, \pi, 2\pi$

c St.pts. $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}\right), \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}e^{\frac{5\pi}{4}}\right)$ Inf. pts. $(0, 1), (\pi, -e^{\pi}), (2\pi, e^{2\pi})$ **d**



9 **a i** $(1 - x)e^{-x}$ **ii** $(x - 2)e^{-x}$ **b i** $x = 1$ **ii** $x = 2$

c St. pt. $(1, e^{-1})$ Inf. pt. $(2, 2e^{-2})$ **d**



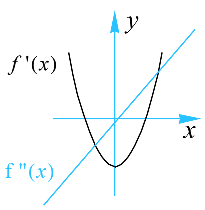
10 **a** 8 **b** 0 **c** 4 **d** $27\sqrt[3]{9} \approx 56.16$

11 **a** min value -82 **b** max value 26

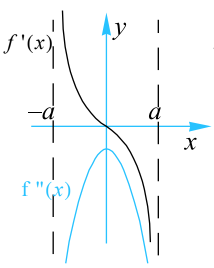
12	**a** pt A:	**i**	Yes	**ii**	non-stationary pt of inflect;
	pt B:	**i**	Yes	**ii**	Stationary point (local/global min);
	pt C:	**i**	Yes	**ii**	non-stationary pt of inflect.
	b pt A:	**i**	No	**ii**.	Local/global max;
	pt B:	**i**	No	**ii**	Local/global min;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	c pt A:	**i**	Yes	**ii**	Stationary point (local/global max);
	pt B:	**i**	Yes	**ii**	Stationary point (local min);
	pt C	**i**	Yes	**ii**	non-stationary pt of inflect.
	d pt A:	**i**	Yes	**ii**	Stationary pt (local/global max);
	pt B:	**i**	No	**ii**	Local min;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	e pt A:	**i**	No	**ii**	Cusp (local min);
	pt B:	**i**	Yes	**ii**	Stationary pt of inflect;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	f pt A:	**i**	Yes	**ii**	Stationary point (local/global max);
	pt B:	**i**	Yes	**ii**	Stationary point (local/global min);
	pt C:	**i**	No	**ii**	Tangent parallel to y–axis.

13 **a i** A **ii** B **iii** C **b i** C **ii** B **iii** A

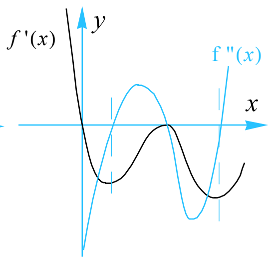
14 **a**



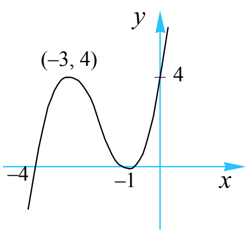
b



c

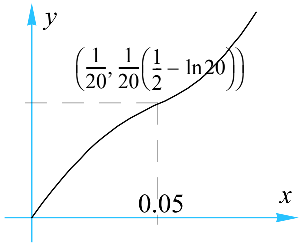


15 $y = x^3 + 6x^2 + 9x + 4$



16 $f(x) = \frac{1}{3}x^3 - x^2 - 3x - 6$

17 $f(x) = 3x^5 - 20x^3$

18 

19 $m = -0.5$, $n = 1.5$

Exercise 6.3.3

1 **a** $4\pi r$ cm^2s^{-1} **b** 4π cms^{-1}

2 6 cm^2s^{-1}

3 **a** $\frac{dA}{dt} = -\frac{3}{2}\sqrt{2}x$ cm^2s^{-1} (x = side length) **b** $-\frac{3}{2}\sqrt{2}$ cms^{-1}

4 **a** 37.5 cm^3h^{-1} **b** 30 cm^2h^{-1} **c** 0.96 $g^{-1}cm^3h^{-1}$

5 ~ 0.37 cms^{-1}

6 −0.24 cm^3min^{-1}

7 **a** 0.035 ms^{-1} **b** 0.035 ms^{-1}

8 8π cm^3min^{-1}

9 854 kmh^{-1}

10 $\frac{53}{6}$

11 2 rad s^{-1}

12 **a** $V = h^2 + 8h$ **b** $\frac{4}{15}$ m min^{-1} **c** 0.56 m^2min^{-1}

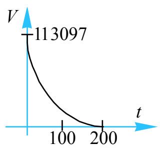
13 $\frac{3\sqrt{10}}{200}$ m min^{-1}

14 $10\sqrt{2}$ cm^3s^{-1}

15 0.9 ms^{-1}

16 −3.92 ms^{-1}

17 **a** $x = 30 - 0.15t$ **b** [0, 200] **c i** 1531 cm^3s^{-1} **ii** 15.90 cm^2s^{-1}

d 

18 ~1.24 ms^{-1}

19 ~0.0696 ms^{-1}

20 **a** $y = \sqrt{119 + 20t - 4t^2}$ **b** ~0.516 ms^{-1}

21 **a** 0.095 cms^{-1} **b** 0.6747 cm^2s^{-1}

22 **a i** $x = 70t$ **ii** $y = 80t$ **b** $130t$ **c** 130 kmh^{-1}

d 14.66 kmh^{-1}

23 -0.77 ms^{-1}

24 0.40 ms^{-1}

25 3.2 ms^{-1}

26 0.075 m min^{-1}

27 1.26° per sec

28 $\frac{5}{2564} \approx 0.002$ rad per second

29 **a** 9% per second **b** 6% per second

30 0.064

31 8211 per year

32 4% per second

33 –0.25 rad per second

Exercise 6.4.1

6. Find the indefinite integral of the following.

a $\sqrt[4]{x^3} + \frac{1}{\sqrt{x}} - 5$ **b** $\sqrt{x}(\sqrt{x} - 2x)(x + 1)$ **c** $\frac{1}{z^3} - \frac{2}{z^2} + 4z + 1$

d $\left(2t + \frac{3}{t^2}\right)\left(t^2 - \frac{1}{t}\right) + \frac{3}{t^3}$ **e** $\frac{(t-2)(t-1)}{\sqrt{t}} - \frac{2}{\sqrt{t}}$ **f** $\frac{u^3 + 6u^2 + 12u + 8}{u + 2}$

7. Given that $f(x) = ax^n$, $n \neq -1$ and $g(x) = bx^m$, $m \neq -1$ show that:

a $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

b $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

c $\int kf(x)dx = k\int f(x)dx$

d $\int [f(x)g(x)]dx \neq \left(\int f(x)dx\right)\left(\int g(x)dx\right)$

e $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

8. **a** Show that $\frac{d}{dx}((2x+3)^4) = 8(2x+3)^3$. Hence find $\int (2x+3)^3 dx$.

b Show that $\frac{d}{dx}(\sqrt{x^2+4}) = \frac{x}{\sqrt{x^2+4}}$. Hence find $\int \frac{3x}{\sqrt{x^2+4}}dx$.

Exercise 6.4.2

9. The acceleration, in m/s² of a body in a medium is given by $\frac{dv}{dt} = \frac{3}{t+1}$, $t \geq 0$. The particle has an initial speed of 6 m/s, find the speed (to 2 d.p) after 10 seconds.

10. The rate of change of the water level in an empty container, t seconds after it started to be filled from a tap is given by the relation:

$$\frac{dh}{dt} = 0.2\sqrt[3]{t+8},\ t \geq 0$$
,

where h cm is the water level. Find the water level after 6 seconds.

11. The gradient function of the curve $y = f(x)$ is given by $e^{0.5x} - \cos(2x)$. Find the equation of the function, given that it passes through the origin.

12. a Given that $\frac{d}{dx}(e^{ax}(p\sin bx + q\cos bx)) = e^{ax}\sin bx$, express p and q in terms of a and b.

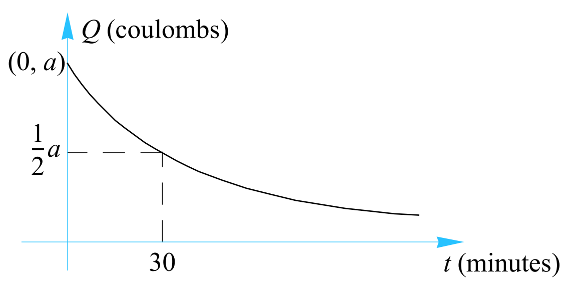
b Hence find $\int e^{2x}\sin 3x dx$.

13 The rate of change of the charge, Q, in coulombs, retained by a capacitor t minutes after charging, is given by $\frac{dQ}{dt} = -ake^{-kt}$.

Using the graph shown, determine the charge remaining after

a one hour

b 80 minutes



14. a Show that $\frac{d}{dx}(x\ln(x+k)) = \frac{x}{x+k} + \ln(x+k)$, where k is a real number.

b For a particular type of commercial fish, it is thought that a length–weight relationship exists such that their rate of change of weight, w kg, with respect to their length, x m, is modelled by the equation:

$$\frac{dw}{dx} = 0.2\ln(x+2).$$

Given that a fish in this group averages a weight of 650 gm when it is 20 cm long, find the weight of a fish measuring 30 cm.

15. The rate of flow of water, $\frac{dV}{dt}$ litres/hour, pumped into a hot water system over a 24-hour period from 6:00 am, is modelled by the relation:

$$\frac{dV}{dt} = 12 + \frac{3}{2}\cos\frac{\pi}{3}t, \ t \geq 0.$$

a Sketch the graph of $\frac{dV}{dt}$ against t.

b For what percentage of the time will the rate of flow exceed 11 litres/hour.

c How much water has been pumped into the hot water system by 8:00 a.m.?

16. The rates of change of the population size of two types of insect pests over a 4-day cycle, where t is measured in days, has been modelled by the equations:

$$\frac{dA}{dt} = 2\pi\cos\pi t, \ t \geq 0 \ \text{ and } \ \frac{dB}{dt} = \frac{3}{4}e^{0.25t}, \ t \geq 0$$

where A and B represent the number of each type of pest in thousands.

Initially there were 5000 insects of type A and 3000 insects of type B.

a On the same set of axes sketch the graphs, $A(t)$ and $B(t)$ for $0 \leq t \leq 4$.

b What is the maximum number of insects of type A that will occur?

c When will there first be equal numbers of insects of both types?

d For how long will the number of type B insects exceed the number of type A insects during the four days?

Exercise 6.4.1

1 **a** $\frac{1}{4}x^4+c$ **b** $\frac{1}{8}x^8+c$ **c** $\frac{1}{6}x^6+c$ **d** $\frac{1}{9}x^9+c$

e $\frac{4}{3}x^3+c$ **f** $\frac{7}{6}x^6+c$ **g** x^9+c **h** $\frac{1}{8}x^4+c$

2 **a** $5x+c$ **b** $3x+c$ **c** $10x+c$ **d** $\frac{2}{3}x+c$

e $-4x+c$ **f** $-6x+c$ **g** $-\frac{3}{2}x+c$ **h** $-x+c$

3 **a** $x-\frac{1}{2}x^2+c$ **b** $2x+\frac{1}{3}x^3+c$ **c** $\frac{1}{4}x^4-9x+c$ **d** $\frac{2}{5}x+\frac{1}{9}x^3+c$

e $\frac{1}{3}x^{3/2}+\frac{1}{x}+c$ **f** $x^{5/2}+4x^2+c$ **g** $\frac{1}{3}x^3+x^2+c$ **h** x^3-x^2+c

i $x-\frac{1}{3}x^3+c$

4 **a** $\frac{1}{3}x^3-\frac{1}{2}x^2-6x+c$ **b** $\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{3}{2}x^2+c$ **c** $\frac{1}{4}(x-3)^4+c$

d $\frac{2}{5}x^5+\frac{1}{2}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+c$ **e** $x+\frac{1}{2}x^2-\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+c$

f $\frac{2}{7}x^{7/2}+\frac{4}{5}x^{5/2}+\frac{2}{3}x^{3/2}-2x+c$

5 **a** $\frac{1}{2}x^2-3x+c$ **b** $2\mathrm{u}^2+5\mathrm{u}+\frac{1}{\mathrm{u}}+c$ **c** $-\frac{1}{x}-\frac{2}{x^2}-\frac{4}{3x^3}+c$

d $\frac{1}{2}x^2+3x+c$ **e** $\frac{1}{2}x^2-4x+c$ **f** $\frac{1}{3}t^3+2t-\frac{1}{t}+c$

6 **a** $\frac{4}{7}\sqrt[4]{x^7}+2\sqrt{x}-5x+c$ **b** $\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{4}{7}x^{7/2}-\frac{4}{5}x^{5/2}+c$

c $-\frac{1}{2z^2}+\frac{2}{z}+2z^2+z+c$ **d** $\frac{1}{2}t^4+t+c$

e $\frac{2}{5}\sqrt{t^5}-2\sqrt{t^3}+c$ **f** $\frac{1}{3}u^3+2u^2+4u+c$

8 a $\frac{1}{8}(2x+3)^4+c$ b $3\sqrt{x^2+4}+c$

Exercise 6.4.2

1 a $\frac{1}{5}e^{5x}+c$ b $\frac{1}{3}e^{3x}+c$ c $\frac{1}{2}e^{2x}+c$

d $10e^{0.1x}+c$ e $-\frac{1}{4}e^{-4x}+c$ f $-e^{-4x}+c$

g $-0.2e^{-0.5x}+c$ h $-2e^{1-x}+c$ i $5e^{x+1}+c$

j $e^{2-2x}+c$ k $3e^{x/3}+\mathrm{c}$ l $2\sqrt{e^x}+c$

2 a $4\log_e x+c, x>0$ b $-3\log_e x+c, x>0$ c $\frac{2}{5}\log_e x+c, x>0$

d $\log_e(x+1)+c, x>-1$ e $\frac{1}{2}\log_e x+c, x>0$ f $x-2\log_e x-\frac{1}{x}+c, x>0$

g $\frac{1}{2}x^2-2x+\log_e x+\mathrm{c}, x>0$ h $3\ln(x+2)+c$

3 a $-\frac{1}{3}\cos(3x)+c$ b $\frac{1}{2}\sin(2x)+c$ c $\frac{1}{5}\tan(5x)+c$ d $\cos(x)+\mathrm{c}$

4 a $-\frac{1}{2}\cos(2x)+\frac{1}{2}x^2+c$ b $2x^3-\frac{1}{4}\sin(4x)+\mathrm{c}$ c $\frac{1}{5}e^{5x}+c$

d $-\frac{4}{3}e^{-3x}-2\cos\left(\frac{1}{2}x\right)+c$ e $3\sin\left(\frac{x}{3}\right)+\frac{1}{3}\cos(3x)+c$

f $\frac{1}{2}e^{2x}+4\log_e x-x+\mathrm{c}, x>0$ g $\frac{1}{2}e^{2x}+2e^{x}+x+c$

h $\frac{5}{4}\cos(4x)+x-\log_e x+c, x>0$ i $\frac{1}{3}\tan(3x)-2\log_e x+2e^{x/2}+c, x>0$

j $\frac{1}{2}e^{2x}-2x-\frac{1}{2}e^{-2x}+c$ k $\frac{1}{2}e^{2x+3}+c$

l $-\frac{1}{2}\cos(2x+\pi)+c$ m $\sin(x-\pi)+c$

n $-4\cos\left(\frac{1}{4}x+\frac{\pi}{2}\right)+c$ **o** $2\left(\frac{e^x+2}{\sqrt{e^x}}\right)+c$

5 **a** $\frac{1}{16}(4x-1)^4+c$ **b** $\frac{1}{21}(3x+5)^7+c$ **c** $-\frac{1}{5}(2-x)^5+c$

d $\frac{1}{12}(2x+3)^6+c$ **e** $-\frac{1}{27}(7-3x)^9+c$ **f** $\frac{1}{5}\left(\frac{1}{2}x-2\right)^{10}+c$

g $-\frac{1}{25}(5x+2)^{-5}+c$ **h** $\frac{1}{4}(9-4x)^{-1}+c$ **i** $-\frac{1}{2}(x+3)^{-2}+c$

j $\ln(x+1)+c, x>-1$ **k** $\ln(2x+1)+c, x>-\frac{1}{2}$ **l** $-2\ln(3-2x)+c, x<\frac{3}{2}$

m $3\ln(5-x)+c, x<5$ **n** $-\frac{3}{2}\ln(3-6x)+c, x<\frac{1}{2}$ **o** $\frac{5}{3}\ln(3x+2)+c, x>-\frac{2}{3}$

6 **a** $-\frac{1}{2}\cos(2x-3)-x^2+c$ **b** $6\sin\left(2+\frac{1}{2}x\right)+5x+c$ **c** $\frac{3}{2}\sin\left(\frac{1}{3}x-2\right)+\ln(2x+1)+c$

d $10\tan(0.1x-5)-2x+c$ **e** $2\ln(2x+3)+2e^{-\frac{1}{2}x+2}+c$ **f** $-\frac{2}{2x+3}-\frac{1}{2}e^{2x-\frac{1}{2}}+c$

g $x+\ln(x+1)-4\ln(x+2)+c$ **h** $2x-3\ln(x+2)+\frac{1}{2}\ln(2x+1)+c$

i $-\frac{1}{2x+1}+\ln(2x+1)+c$

7 **a** $f(x)=\frac{1}{6}\sqrt{(4x+5)^3}$ **b** $f(x)=2\ln(4x-3)+2$

c $f(x)=\frac{1}{2}\sin(2x+3)+1$ **d** $f(x)=2x+\frac{1}{2}e^{-2x+1}+\frac{1}{2}e$

8 14 334

9 13.19ms^{-1} or 1.19ms^{-1}

10 2.66 cm

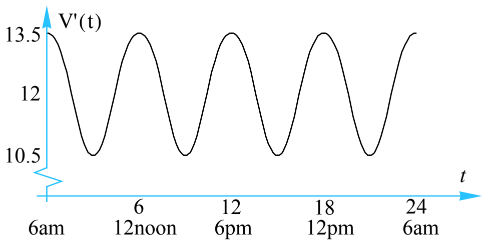
11 $2e^{x/2}-\frac{1}{2}\sin(2x)-2$

12 **a** $p = \frac{a}{a^2+b^2}$, $q = -\frac{b}{a^2+b^2}$ **b** $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + c$

13 **a** $0.25a$ **b** $a \times \left(\frac{1}{2}\right)^{8/3} \approx 0.1575a$

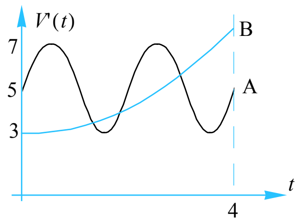
14 **b** 666 g

15 **a**



b 73.23% **c** ~25.24 litres

16 **a**

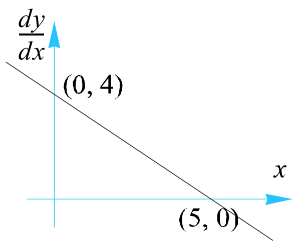


b 7000 **c** 1.16 day **d** 2 days

Exercise 6.5.1

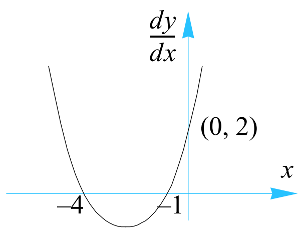
11. Sketch the graph of $y = f(x)$ for each of the following:

a



Where the curve passes through the point (5, 10).

b



Where the curve passes through the point (0, 0).

12. Find $f(x)$ given that $f''(x) = 12x + 4$ and that the gradient at the point (1, 6) is 12.

13. Find $f(x)$ given that $f'(x) = ax^2 + b$, where the gradient at the point (1, 2) is 4, and that the curve passes through the point (3, 4).

14. The rate at which a balloon is expanding is given by

$$\frac{dV}{dt} = kt^{4.5}, t \geq 0,$$

where t is the time in minutes since the balloon started to be inflated and V cm^3 is its volume. Initially the balloon (which may be assumed to be spherical) has a radius of 5 cm. If the balloon has a volume of 800 cm^3 after 2 minutes, find its volume after 5 minutes.

15. The area, A cm^2, of a healing wound caused by a fall on a particular surface decreases at a rate given by the equation:

$$A'(t) = -\frac{35}{\sqrt{t}}$$

where t is the time in days. Find the initial area of such a wound if after one day the area measures 40 cm^2.

Exercise 6.5.2

6. Evaluate the following definite integrals (giving exact values).

g $\displaystyle\int_0^1 \frac{2}{(x+1)^3}dx$ h $\displaystyle\int_2^4 \left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2 dx$ i $\displaystyle\int_3^4 \frac{2x+1}{2x^2-3x-2}dx$

8. Evaluate the following definite integrals (giving exact values).

e $\displaystyle\int_0^{\frac{\pi}{4}} (x-\sec^2 x)dx$ f $\displaystyle\int_0^{\frac{\pi}{2}} 2\cos\left(4x+\frac{\pi}{2}\right)dx$ g $\displaystyle\int_{-\pi}^{\pi} \left(\sin\left(\frac{x}{2}\right)+2\cos(x)\right)dx$

h $\displaystyle\int_0^{\frac{\pi}{12}} \sec^2\left(\frac{\pi}{4}-2x\right)dx$ i $\displaystyle\int_0^{\pi} \cos(2x+\pi)dx$

13. a Find $\frac{d}{dx}(xe^{0.1x})$. Hence, find $\int xe^{0.1x}dx$.

b Following an advertising initiative by the Traffic Authorities, preliminary results predict that the number of alcohol-related traffic accidents has been decreasing at a rate of $-12-te^{0.1t}$ accidents per month, where t is the time in months since the advertising campaign started.

i How many accidents were there over the first six months of the campaign?

ii In the year prior to the advertising campaign there were 878 alcohol-related traffic accidents. Find an expression for the total number of accidents since the start of the previous year, t months after the campaign started.

14. The rate of cable television subscribers in a city t years from 1995 has been modelled by the equation $\dfrac{2000}{\sqrt{(1+0.4t)^3}}$.

a How many subscribers were there between 1998 and 2002?

b If there were initially 40 000 subscribers, find the number of subscribers by 2010.

15.

a Find $\dfrac{d}{dt}\left(\dfrac{800}{1+24e^{-0.02t}}\right)$.

b The rate at which the number of fruit flies appear when placed in an environment with limited food supply in an experiment was found to be approximated by the exponential model:

$$\frac{384e^{-0.02t}}{(1+24e^{-0.02t})^2},\ t \geq 0,$$

where t is the number of days since the experiment started. What was the increase in the number of flies after 200 days?

Exercise 6.5.3

20. Find the area of the region bounded by the curves with equations $y = \sqrt{x}$, $y = 6 - x$ and the x-axis.

21. a Sketch the graph of the function $f(x) = |e^x - 1|$.

b Find the area of the region enclosed by the curve $y = f(x)$,

i the x-axis and the lines $x = -1$ and $x = 1$.

ii the y-axis and the line $y = e - 1$.

iii and the line $y = 1$. Discuss your findings for this case.

22. a On the same set of axes, sketch the graphs of $f(x) = \sin\left(\frac{1}{2}x\right)$ and $g(x) = \sin 2x$ over the interval $0 \le x \le \pi$.

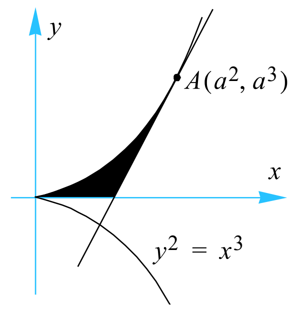
b Find the area of the region between by the curves $y = f(x)$ and $y = g(x)$ over the interval $0 \le x \le \pi$, giving your answer correct to two decimal places.

23. Consider the curve with equation $y^2 = x^3$ as shown in the diagram.

A tangent meets the curve at the point $A(a^2, a^3)$.

a Find the equation of the tangent at A.

b Find the area of the shaded region enclosed by the curve, the line $y = 0$ and the tangent.



24. a On a set of axes, sketch the graph of the curve $y = e^{x-1}$ and find the area of the region enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$.

b Hence evaluate $\int_{e^{-1}}^{1} (\ln x + 1)\,dx$.

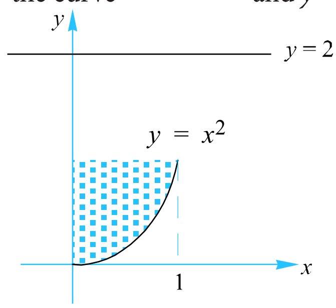
c Find the area of the region enclosed by the curves $y = e^{x-1}$ and $y = \ln x + 1$ over the $e^{-1} \le x \le 1$.

Exercise 6.5.4

Extension problems

23. Find the volume of the solid of revolution generated when the shaded region shown below is revolved about the line $y=2$.

24. Find the volume of the solid of revolution generated by revolving the region enclosed by the curve $y = 4 - x^2$ and $y = 0$ about:



a the line $y = -3$

b the line $x = 3$

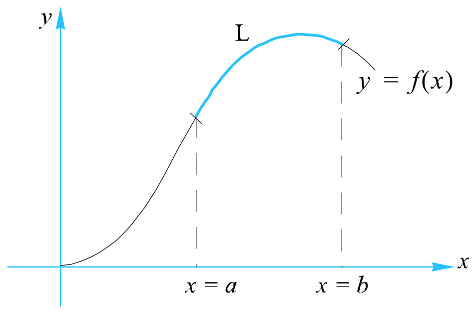
c the line $y = 7$.

In each case, draw the shape of the solid of revolution.

25. Show why the arc length, L units, of a curve from:

$x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx .$$



Exercise 6.5.1

1 a x^2+x+3 b $2x-\frac{1}{3}x^3+1$ c $\frac{8}{3}\sqrt{x^3}-\frac{1}{2}x^2-\frac{40}{3}$

d $\frac{1}{2}x^2+\frac{1}{x}+2x-\frac{3}{2}$ e $(x+2)^3$ f $\frac{3}{4}\sqrt[3]{x^4}+\frac{1}{4}x^4+x$

g $\frac{1}{3}x^3+1$ h x^4-x^3+2x+3

2 $\frac{1}{2}x^2+\frac{1}{x}+\frac{5}{2}$

3 \$3835.03

4 9.5

5 $\frac{251}{3}\pi$ cm^3

6 292

7 $\frac{5}{7}\sqrt{x^3}+\frac{23}{7}$

8 1, −8

9 $P(x) = 25-5x+\frac{1}{3}x^2$

10 $N = \frac{20000}{201}t^{2.01}+500, t\geq 0$

11 a $y = -\frac{2}{5}x^2+4x$ b $y = \frac{1}{6}x^3+\frac{5}{4}x^2+2x$

12 $y = 2(x^3+x^2+x)$

13 $f(x) = -\frac{3}{10}x^3+\frac{49}{10}x-\frac{13}{5}$

14 Vol ~ 43 202 cm^3

15 110 cm^2

Exercise 6.5.2

1 a $\frac{15}{2}$ b $\frac{38}{3}$ c $\frac{5}{36}$ d −8

2 a $\frac{35}{24}$ b $\frac{8}{5}\sqrt{2}-2$ c −2 d 0

e $\frac{1}{20}$ f $-\frac{4}{3}$ g $\frac{7}{6}$ h $\frac{5}{6}$

i $\frac{20}{3}$ j 0 k $\frac{20}{3}$ l $-\frac{\sqrt{2}}{3}$

4	**a**	e	**b**	$2(e^{-2}-e^{-4})$	**c**	0	**d**	$2(e-e^{-1})$	
	e	e^2+4-e^{-2}	**f**	$\frac{1}{2}(e-e^5)$	**g**	$2\sqrt{e}-3$			
	h	$\frac{1}{4}(16e^{1/4}-e^4-15)$			**i**	$\frac{1}{2}(e^{-1}-e^3)$			
6	**a**	$3\ln 2$	**b**	$2\ln 5$	**c** $4+4\ln 3$		**d**	$\frac{1717}{4}$	
	e	$\frac{3}{2}\ln 3$	**f**	$2\ln 2$	**g**	$\frac{3}{4}$	**h**	$4\ln 2-2$	**i** $\ln 2$
8	**a**	1	**b**	$\frac{3\sqrt{3}}{2}$	**c** $\frac{\sqrt{3}}{2}$		**d**	-2	
	e	$\frac{\pi^2}{32}-1$	**f**	0	**g**	0	**h**	$\frac{\sqrt{3}}{2}-\frac{1}{2}$	
	i	0	**j**	2					
9	**a**	$\frac{31}{5}$	**b**	$\frac{7\sqrt{7}}{3}-\sqrt{3}$	**c**	0	**d**	$\frac{5}{72}$	
	e	$3\sqrt[3]{2}-\frac{3}{2}$	**f**	$1-\ln 2$	**g**	$\frac{76}{15}$	**h**	$\frac{16}{15}$	
	i	$\frac{2}{3}(e+1)^{3/2}(1-e^{-3/2})$							

10 $\ln\left(\frac{21}{5}\right)$

11 $\sin 2x+2x\cos 2x$; 0

12	**a**	$2m-n$	**b**	$m+a-b$	**c** $-3n$		**d**	$m(2a-b)$	**e** na^2

13 **a** $e^{0.1x}+0.1xe^{0.1x}$; $10xe^{0.1x}-100e^{0.1x}+c$

b i 99 accidents **ii** $N = 12t+10te^{0.1t}-100e^{0.1t}+978$

14 **a** 1612 subscribers **b** 46 220

15 **b** ~524 flies

Exercise 6.5.3

1 **a** 4 sq.units **b** $\frac{32}{3}$ sq.units **c** 4 sq.units
d 36 sq.units **e** $\frac{1}{6}$ sq.units

2 **a** e sq.units **b** $\frac{1}{2}(e^4-2-e^2)$ sq.units **c** $2(e+e^{-1}-2)$ sq.units
d $2(e^2-2-e)$ sq.units

3 **a** $\ln\left(\frac{5}{4}\right)$ sq.units **b** 2ln5 sq.units **c** 3ln3 sq.units **d** 0.5 sq.units

4 **a** 2 sq.units **b** $\frac{\pi}{2}$ sq.units **c** $\frac{3}{8}\pi^2+\sqrt{2}-2$ sq.units
d $\sqrt{2}$ sq. units **e** $4\sqrt{3}$ sq.units

6 12 sq. units

7 $4\left(\sqrt{3}-\frac{1}{3}\right)$ sq.units.

8 ln2 + 1.5 sq.units.

9 2 sq.units.

10 $\frac{37}{12}$ sq. units

11 **a** 0.5 sq. units **b** 1 sq. unit **c** $2(\sqrt{6}-\sqrt{2})$ sq. units

12 $\frac{8}{3}$

13 $-2\tan 2x$; $\frac{1}{4}\ln 2$ sq.units

14 **a** $\frac{9}{2}$ sq. units **b** 3 sq. units

15 **a** 1 sq.unit **b** 10 sq. units

16 **a** $x\ln x - x + c$ **b** 1 sq. unit

17 $\frac{14}{3}$ sq. units

18 **a** $\frac{7}{6}$ sq. units **b** $\frac{9}{2}$ sq. units

19 **a i** $\frac{15}{4}$ sq. units **ii** $\frac{45}{4}$ sq. units

20 $\frac{22}{3}$ sq. units

21	**b i**	$e^{-1}+e-2$ sq. units	**ii**	1 sq. unit	**iii**	2ln(2) sq. units
22	**b**	3.05 sq. units				
23	**a**	$2y = 3ax - a^3$	**b**	$\frac{1}{15}a^5$ sq. units		
24	**a**	$1-e^{-1}$ sq. units	**b**	e^{-1} sq. units	**c**	$1-e^{-e^{-1}-1}-e^{-1}$ ~ 0.10066 sq. units

Exercise 6.5.4

All values are in cubic units.

1 21π

2 *p*ln5

3 $\frac{4}{5}\pi$

4 $\frac{\pi}{2}(e^{10}-e^2)$

5 π^2

6 $\frac{\pi}{2}$

7 $\frac{109}{3}\pi$

8 $\pi\left(\frac{8}{3}-2\ln 3\right)$

12 $\frac{\pi}{2}(5-5\sin 1)$

13 $\frac{251}{30}\pi$

14 **a** 40π **b** $\frac{242}{5}\pi$

15 **a** $\frac{8}{35}\pi$ **b** $\frac{\pi}{4}$

16 **a** $\frac{9}{2}\pi$ **b** $\frac{88}{5}\sqrt{3}\pi$

17 $\frac{3\pi}{4}$

18 $k = 1$

19 $4\pi^2a^2$

20 $k = \frac{\pi}{2}$

21 **b i** $\frac{\pi a}{2(1+a^2)}$ **ii** $\frac{8\pi}{15}\sqrt{\frac{a}{1+a^2}}\left(\frac{3a^2+2}{1+a^2}\right)$

22 **a** Two possible solutions: solving $a^3-6a^2-36a+204 = 0$, a = 4.95331; solving $a^3-6a^2-36a-28 = 0$, then a = –0.95331

b $a = \frac{100}{\pi}$

23 $\frac{28}{15}\pi$

24. **a** $\frac{1472}{15}\pi$ **b** 64π **c** $\frac{576}{5}\pi$

Exercise 6.6.1

1 **a** $x = t^3 + 3t + 10, t \geq 0$ **b** $x = 4\sin t + 3\cos t - 1, t \geq 0$ **c** $x = t^2 - 4e^{-\frac{1}{2}t} + 2t + 4, t \geq 0$

2 **a** $x = t^3 - t^2, t \geq 0$ **b** 100 **c** $100\frac{8}{27}$ m

3 **a** $x = -\frac{2}{3}(4+t)^{3/2} + 2t + 8$ **b** 6.92 m

4 $\frac{125}{6}$ m

5 $\frac{125}{49}$ s; 63.8 m

6 **a** $\frac{\pi}{6}$ s **b** $\frac{\pi}{2} - 1$ m

7 80.37 m

8 **a** $s(t) = \frac{160}{\pi}\left[1 - \cos\left(\frac{\pi}{16}t\right)\right], t \geq 0$ **b** 86.94 m

c –6.33 m **d** 116.78 m

9 **a** $v = 4 + k - \frac{k}{t^2}, t > 0$ **b** $k = 2$ **c** 52.2 m

10 **b** 0.0893 m

Exercise 6.7.1

For this set of exercises, use the method of recognition to determine the integrals.

1. Find the following indefinite integrals.

g $\int \frac{4x}{\sqrt{x^2+2}}dx$ h $\int \frac{x^3}{(1-x^4)^4}dx$ i $\int 3e^{3x}\sqrt{1+e^{3x}}dx$

j $\int \frac{x+1}{(x^2+2x-1)^2}dx$ k $\int \frac{x^2+1}{\sqrt{x^3+3x+1}}dx$ l $\int x\sqrt{3+4x^2}dx$

m $\int \frac{e^x}{\sqrt{e^x+2}}dx$ n $\int \frac{e^{-2x}}{(1-e^{-2x})^3}dx$ o $\int 10x^2(x^3+1)^4dx$

p $\int (x^3+2)(x^4+8x-3)^5dx$ q $\int 2x^3\sqrt{(x^4+5)^3}dx$ r $\int \frac{\cos 2x}{\sqrt{1-\sin 2x}}dx$

s $\int \cos x\sqrt{4+3\sin x}dx$ t $\int \frac{\sec^2 4x}{(1+3\tan 4x)^2}dx$ u $\int \frac{1-\sin x}{\sqrt[3]{x+\cos x}}dx$

v $\int \sin\frac{1}{2}x\cos^3\frac{1}{2}x dx$ w $\int \frac{x\cos x+\sin x}{\sqrt{1+x\sin x}}dx$ x $\int \frac{(\sqrt{x}+1)^{1/2}}{\sqrt{x}}dx$

2. Find the antiderivative of the following.

g $e^x\sin(2e^x)$ h $\frac{e^{2x}}{(1-e^{2x})^2}$ i $\frac{e^{-x}}{1+e^{-x}}$

j $\frac{5}{e^{-x}+2}$ k $e^{-ax}\sqrt{4+e^{-ax}}$ l $\frac{e^{2x}}{1+e^{2x}}\ln(1+e^{2x})$

3. Find the antiderivative of the following.

g $\frac{4\sec^2 3x}{(1+\tan 3x)^2}$ h $\frac{2}{x}\cos(\ln x)$ i $\sin x\cos x\sqrt{1+\cos 2x}$

j $e^x\cos(e^x)$ k $3x^2e^{-x^3+2}$ l $\cot\frac{1}{2}x\ln\left(\sin\frac{1}{2}x\right)$

m $\sin x\sec^2 x$ n $\frac{1}{e^{-x}+2}\ln(1+2e^x)$ o $(x^2-3)\sec^2\left(\frac{1}{3}x^3-3x\right)$

4. Find the antiderivative of the following.

e $\frac{1}{\sqrt{16-x^2}}$

f $-\frac{1}{\sqrt{9-x^2}}$

5. Find the following indefinite integrals.

e $\int \frac{1}{\sqrt{1-4x^2}}dx$

f $\int \frac{1}{\sqrt{9-4x^2}}dx$

g $\int \frac{1}{\sqrt{4-25x^2}}dx$

h $\int \frac{2}{1+4x^2}dx$

i $\int \frac{1}{9+4x^2}dx$

j $\int \frac{1}{9+16x^2}dx$

k $\int \frac{1}{9+5x^2}dx$

l $\int \frac{1}{\sqrt{3-5x^2}}dx$

6. Evaluate:

k $\int_{-2}^{2} \frac{x}{\sqrt{9-x^2}}dx$

l $\int_{1}^{2} \frac{4x}{(1+x^2)^2}dx$

m $\int_0^1 \frac{1}{x^2+1}dx$

n $\int_{-1}^{1} \frac{1}{4+x^2}dx$

o $\int_0^3 \frac{1}{1+9x^2}dx$

p $\int_0^{1/3} \frac{1}{\sqrt{1-4x^2}}dx$

q $\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}dx$

r $\int_0^{\frac{1}{2}} \frac{1}{9x^2+1}dx$

s $\int_{\frac{1}{8}}^{\frac{1}{2}} \frac{1}{4+9x^2}dx$

t $\int_0^{\frac{\pi}{6}} \sin^3 x \cos x dx$

u $\int_0^{\frac{\pi}{4}} \sec x \tan^3 x dx$

v $\int_1^{e} \frac{(\ln x)^2}{x}dx$

w $\int_0^{1} \frac{2-x^2}{\sqrt{1-x^2}}dx$

x $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 3x}{(4+\sin 3x)^2}dx$

Example

Evaluate the following:

a $\displaystyle\int_{-1}^{2} \frac{1}{x^2+2x+5}dx$ b $\displaystyle\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{2+4x-x^2}}dx$.

Presented with the definite integral $\int_{-1}^{2} \frac{1}{x^2+2x+5}dx$, the first thing we observe is that the denominator is nearly a perfect square (sort of).

In fact, $x^2+2x+5 = (x+1)^2+4$.

So, we have $\int_{-1}^{2} \frac{1}{x^2+2x+5}dx = \int_{-1}^{2} \frac{1}{(x+1)^2+4}dx$, which is in the form suitable for an inverse tangent function.

Having recognized it as a Tan^{-1} you can make use of the substitution, $x+1 = 2\tan(\theta)$.

So, letting $x+1 = 2\tan(\theta)$ we have $\frac{dx}{d\theta} = 2\sec^2(\theta)$, when $x = -1$, $\theta = 0$ and when $x = 2$, $\theta = \frac{\pi}{4}$.

This then gives: $\displaystyle\int_{-1}^{2} \frac{1}{(x+1)^2+4}dx = \int_{0}^{\frac{\pi}{4}} \frac{1}{4\tan^2\theta+4} 2\sec^2(\theta)d\theta$.

$$= \int_{0}^{\frac{\pi}{4}} \frac{2}{4(\tan^2\theta+1)} \sec^2(\theta)d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{2\sec^2(\theta)} \sec^2(\theta)d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}d\theta$$

$$= \left[\frac{1}{2}\theta\right]_{0}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

This time we have $\displaystyle\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{5+4x-x^2}}dx$ which is indicative of a Sin^{-1} function (after some manipulation!)

We start with some rearranging: $\int_2^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x^2-4x+4)}}dx = \int_2^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x-2)^2}}dx$.

This time, letting $u = x-2$, we recognize it as an inverse sine function of the form

$$\int \frac{1}{\sqrt{9-u^2}}du \text{ (with } u = x-2\text{)}.$$

So, $\int_2^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x-2)^2}}dx = \frac{1}{3}\int_2^{2+\sqrt{3}} \frac{3}{\sqrt{9-(x-2)^2}}dx = \frac{1}{3}\left[\text{Sin}^{-1}\left(\frac{x-2}{3}\right)\right]_2^{2+\sqrt{3}}$

$$= \frac{1}{3}\left[\text{Sin}^{-1}\left(\frac{2+\sqrt{3}-2}{3}\right) - \text{Sin}^{-1}\left(\frac{2-2}{3}\right)\right] = \frac{1}{3}\text{Sin}^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

Exercise 6.7.2

1. Find the following, using the given u substitution.

h $\int t \cdot \sqrt[3]{2-t^2}\,dt$, $u = 2-t^2$

i $\int \cos x e^{\sin x}dx$, $u = \sin x$

j $\int \frac{e^x}{e^x+1}dx$, $u = e^x+1$

k $\int \cos x \sin^4 x\,dx$, $u = \sin x$

l $\int x\sqrt{x+1}\,dx$, $u = x+1$

2. Using the substitution method, find:

e $\int \frac{4x}{(1-2x^2)}dx$

f $\int \frac{4x}{(1-2x^2)^2}dx$

g $\int \frac{1}{x}(\log_e x)dx$

h $\int \frac{e^{-x}}{1+e^{-x}}dx$

i $\int \frac{1}{x\log_e x}dx$

3. Using an appropriate substitution, evaluate the following, giving exact values.

e $\int_0^{\frac{\pi}{12}} \frac{\sec^2 3x}{1+\tan 3x}dx$

f $\int_0^1 \frac{x}{(1+x^2)^2}dx$

g $\int_0^1 4x\sqrt{4+5x^2}\,dx$

h $\int_1^2 x\sqrt{x-1}\,dx$

i $\int_{-1}^1 e^x\sqrt{e^x+1}\,dx$

4. Using an appropriate substitution, evaluate the following, giving exact values.

e $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}dx$

f $\displaystyle\int_1^2 5xe^{(2x^2-3)}dx$

g $\displaystyle\int_{-1}^1 (3-2x)^7 dx$

h $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{(1-x)\sqrt{1-x^2}}dx$

i $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin 2x}dx$

5. Using an appropriate substitution, find the following, giving exact values where required.

e $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 x}dx$

f $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x dx$

6. Using an appropriate substitution, find the following, giving exact values where required.

c $\displaystyle\int_3^6 \frac{x}{\sqrt{x-2}}dx$

d $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{x+1}}dx$

e $\displaystyle\int_{-2}^0 (x-2)\sqrt{x+2}dx$

f $\displaystyle\int_2^5 \frac{x+1}{x-1}dx$

7. Find the following indefinite integrals.

e $\displaystyle\int \frac{2}{\sqrt{5+3x-x^2}}dx$

f $\displaystyle\int \frac{x}{\sqrt{9-x^4}}dx$

g $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

h $\displaystyle\int \frac{(\arccos x)^2}{\sqrt{1-x^2}}dx$

i $\displaystyle\int \frac{1}{\arcsin^3 x\sqrt{1-x^2}}dx$

13. Evaluate the following definite integrals.

a $\displaystyle\int_0^1 \frac{2}{\sqrt{4-u^2}}du$

b $\displaystyle\int_{-2}^2 \frac{4}{\sqrt{9-x^2}}dx$

c $\displaystyle\int_0^{1/4} \frac{3}{\sqrt{1-4x^2}}dx$

d $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^4}}dx$

e $\int_{0}^{1} \frac{2x-2}{\sqrt{2-x^2}}dx$

f $\int_{-1}^{1} \frac{2}{4+(x+1)^2}du$

g $\int_{-3}^{0} \frac{2}{x^2+6x+10}du$

Exercise 6.7.3

1. Integrate the following expressions with respect to x.

e $5xe^{-4x}$ f $\ln x$ g $x\ln x$ h $-x\cos(-5x)$

i $4x\sin\left(-\frac{x}{3}\right)$ j $\frac{x}{\cos^2 x}$ k $\sqrt{x}\ln x$

2. Use integration by parts to antidifferentiate:

c $(x+1)\sqrt{x+2}$

3. Find:

c $\int \text{Sin}^{-1}x\,dx$

4. Find:

c $\int x\,\text{Sin}^{-1}x\,dx$

Exercise 6.7.1

1

a $\frac{2}{3}(5x^2+2)^{3/2}+c$ **b** $-\frac{1}{3(x^3+4)}+c$ **c** $\frac{3}{8}(1-2x^2)^4+c$

d $\frac{1}{5}(9+2x^{3/2})^5+c$ **e** $\frac{9}{4}(x^2+4)^{4/3}+c$ **f** $\frac{-1}{2(x^2+3x+1)^2}+c$

g $4\sqrt{x^2+2}+c$ **h** $\frac{1}{12(1-x^4)^3}+c$ **i** $\frac{2}{3}(1+e^{3x})^{3/2}+c$

j $\frac{-1}{2(x^2+2x-1)}+c$ **k** $\frac{2}{3}\sqrt{x^3+3x+1}+c$ **l** $\frac{1}{12}(3+4x^2)^{3/2}+c$

m $2\sqrt{e^x+2}+c$ **n** $-\frac{1}{4}(1-e^{-2x})^{-2}+c$ **o** $\frac{2}{3}(x^3+1)^5+c$

p $\frac{1}{24}(x^4+8x-3)^6+c$ **q** $\frac{1}{5}(x^4+5)^{5/2}+c$ **r** $-\sqrt{1-\sin 2x}+c$

s $\frac{2}{9}(4+3\sin x)^{3/2}+c$ **t** $-\frac{1}{12(1+3\tan 4x)}+c$ **u** $\frac{3}{2}(x+\cos x)^{2/3}+c$

v $-\frac{1}{2}\cos^4\frac{x}{2}+c$ **w** $2\sqrt{1+x\sin x}+c$ **x** $\frac{4}{3}(x^{1/2}+1)^{3/2}+c$

2

a $e^{x^2+1}+c$ **b** $6e^{\sqrt{x}}+c$ **c** $\frac{1}{3}e^{\tan 3x}+c$

d $-e^{-(ax^2+bx)}+c$ **e** $-6e^{\cos\frac{x}{2}}+c$ **f** $-4e^{(4+x^{-1})}+c$

g $-\frac{1}{2}\cos(2e^x)+c$ **h** $\frac{1}{2(1-e^{2x})}+c$ **i** $-\ln(1+e^{-x})+c$

j $\frac{5}{2}\ln(1+2e^x)+c$ **k** $-\frac{2}{3a}(4+e^{-ax})^{3/2}+c$ **l** $\frac{(\ln(1+e^{2x}))^2}{4}+c$

3

a $-\cos(x^2+1)+c$ **b** $-10\cos\sqrt{x}+c$ **c** $-2\sin\left(2+\frac{1}{x}\right)+c$

d $-\frac{2}{3}(\cos x)^{3/2}+c$ **e** $-\frac{1}{3}\log(\cos 3x)+c$ **f** $\frac{4}{3}\log(1+\tan 3x)+c$

g $\frac{-4}{3(\tan(3x)+1)}+c$ **h** $2\sin(\ln x)+c$ **i** $-\frac{1}{6}(1+\cos 2x)^{3/2}+c$

j $\sin(e^x)+c$ **k** $-e^{(-x^3+2)}+c$ **l** $\left[\ln\left(\sin\frac{1}{2}x\right)\right]^2+c$

m $\sec x+c$ **n** $\frac{1}{4}[\ln(1+2e^x)]^2+c$ **o** $\tan\left(\frac{1}{3}x^3-3x\right)+c$

4

a $\mathrm{Tan}^{-1}\left(\frac{x}{2}\right)+c$ **b** $\mathrm{Tan}^{-1}\left(\frac{x}{3}\right)+c$ **c** $\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

d $\mathrm{Sin}^{-1}\left(\frac{x}{5}\right)+c$ **e** $\mathrm{Sin}^{-1}\left(\frac{x}{4}\right)+c$ **f** $\mathrm{Cos}^{-1}\left(\frac{x}{3}\right)+c$

5

a $3\mathrm{Tan}^{-1}x+c$ **b** $5\mathrm{Sin}^{-1}x+c$ **c** $\mathrm{Sin}^{-1}\left(\frac{x}{2}\right)+c$

d $\mathrm{Sin}^{-1}\left(\frac{x}{3}\right)+c$ **e** $\frac{1}{2}\mathrm{Sin}^{-1}2x+c$ **f** $\frac{1}{2}\mathrm{Sin}^{-1}\left(\frac{2x}{3}\right)+c$

g $\frac{1}{5}\mathrm{Sin}^{-1}\left(\frac{5x}{2}\right)+c$ **h** $\mathrm{Tan}^{-1}2x$ **i** $\frac{1}{6}\mathrm{Tan}^{-1}\left(\frac{2x}{3}\right)+c$

j $\frac{1}{12}\mathrm{Tan}^{-1}\left(\frac{4x}{3}\right)+c$ **k** $\frac{\sqrt{5}}{15}\mathrm{Tan}^{-1}\left(\frac{\sqrt{5}x}{3}\right)+c$ **l** $\frac{1}{\sqrt{5}}\mathrm{Sin}^{-1}\left(\sqrt{\frac{5}{3}}x\right)+c$

6

a $\frac{531377}{9}$ **b** $-2\sqrt{2}+2\sqrt{1+e}$ **c** $3\ln 2(2+\sqrt{2})$

d $4\mathrm{Tan}^{-1}\left(\frac{\pi}{2}\right)$ **e** $\sin e-\sin(e^{-1})$ **f** $\frac{2}{3}\left[1-\cos\left(\frac{\pi}{2}\right)^{3/2}\right]$

g $\frac{2}{3}$ **h** $e-e^{-1}$ **i** $\ln 2$

j $\frac{7\sqrt{7}}{3}$ **k** 0 **l** $\frac{3}{5}$

m $\frac{\pi}{4}$ **n** $\frac{\pi}{2}-\text{Tan}^{-1}(2)$ **o** $\frac{1}{3}\text{Tan}^{-1}9$

p $\frac{1}{2}\text{Sin}^{-1}\left(\frac{2}{3}\right)$ **q** $\frac{1}{4}\left(\pi-2\text{Sin}^{-1}\left(\frac{2}{3}\right)\right)$ **r** $\frac{1}{3}\text{Tan}^{-1}\left(\frac{3}{2}\right)$

s $\frac{1}{6}\left(\text{Tan}^{-1}\left(\frac{3}{4}\right)-\text{Tan}^{-1}\left(\frac{3}{16}\right)\right)$ **t** $\frac{1}{64}$ **u** $\frac{2}{3}-\frac{\sqrt{2}}{3}$

v $\frac{1}{3}$ **w** $\frac{3\pi}{4}$ **x** $-\frac{1}{60}$

Exercise 6.7.2

1

a $\frac{2}{3}(x^2+1)^{3/2}+c$ **b** $\frac{2}{3}(x^3+1)^{3/2}+c$ **c** $-\frac{1}{3}(4-x^4)^{1.5}+c$

d $\ln(x^3+1)+c$ **e** $-\frac{1}{18(3x^2+9)^3}+c$ **f** $e^{(x^2+4)}+c$

g $\ln(z^2+4z-5)+c$ **h** $-\frac{3}{8}(2-t^2)^{4/3}+c$ **i** $e^{\sin x}+c$

j $\ln[e^x+1]+c$ **k** $\frac{1}{5}\sin^5 x+c$ **l** $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$

2

a $\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+c$ **b** $-\frac{2}{3}(1-x)^{3/2}+\frac{4}{5}(1-x)^{5/2}-\frac{2}{7}(1-x)^{7/2}+c$

c $\frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+c$ **d** $e^{\tan x}+c$

e $-\ln(1-2x^2)+c$ **f** $\frac{1}{1-2x^2}+c$

g $\frac{1}{2}(\ln x)^2+c$ **h** $-\ln(1+e^{-x})+c$

i $\ln(\ln x)+c$

3

a 0 **b** $\frac{2\ln 2}{3}$ **c** $\ln\frac{77}{54}$

d $\ln 2$ **e** $\frac{1}{3}\ln 2$ **f** $\frac{1}{4}$

g $\frac{76}{15}$ **h** $\frac{16}{15}$ **i** $\frac{2}{3}(1+e)^{3/2}(1-e^{-3/2})$

4

a $\frac{7\sqrt{7}}{3}-\frac{8}{3}$ **b** $\frac{3}{8}(\cos\pi^2-1)$ **c** $\frac{1042}{5}$

d $\ln 4$ **e** 1 **f** $\frac{5}{4}(e^5-e^{-1})$

g $24\,414$ **h** $\sqrt{3}-\sqrt{2}$ **i** $\frac{1}{4}\ln 3$

5

a $\frac{1}{4}$ **b** $2-\frac{2}{3}\sqrt{3}$ **c** $\frac{31}{80}$

d $4-2\sqrt{2}$ **e** $\ln 2$ **f** $\frac{2}{3}$

6

a $-\frac{2}{5}\sqrt{3}$ **b** $\frac{2}{5}\sqrt{3}$ **c** $\frac{26}{3}$

d $-\frac{4}{3}$ **e** $-\frac{56}{15}\sqrt{2}$ **f** $3+2\ln 4$

7

a $\mathrm{Tan}^{-1}(x+3)+c$ **b** $\frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$ **c** $\mathrm{Sin}^{-1}\left(\frac{x-2}{\sqrt{5}}\right)+c$

d $3\mathrm{Sin}^{-1}\left(\frac{x+1}{3}\right)+c$ **e** $2\sin^{-1}\left(\frac{2x-3}{\sqrt{29}}\right)+c$ **f** $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+c$

g $\frac{1}{2}(\arcsin x)^2+c$ **h** $-\frac{1}{3}(\arccos x)^3+c$ **i** $-\frac{1}{2}(\arcsin x)^{-2}+c$

8 **a** $A = 1, B = -2$

9 **a** $\text{Tan}^{-1}k$ **b i** $\frac{\pi}{6}$ **ii** $\frac{\pi}{4}$ **c** $\frac{\pi}{2}, \pi$

10 $2\sqrt{x} - 2\ln(\sqrt{x} + 1), 2 - 2\ln 2$

11 $\frac{3k^2\pi}{8}$

12

a $\frac{\pi}{3}$ **b** $8\text{Sin}^{-1}\left(\frac{2}{3}\right)$ **c** $\frac{\pi}{4}$

d $\frac{1}{2}\text{Sin}^{-1}(1)$ **e** $2\sqrt{2} - 2 - \frac{\pi}{2}$ **f** $\frac{\pi}{4}$

g $\pi - 2\text{Tan}^{-1}\left(\frac{1}{3}\right)$

Exercise 6.7.3

1

a $\sin x - x\cos x + c$ **b** $4\cos\frac{x}{2} + 2x\sin\frac{x}{2} + c$ **c** $2\left(4\sin\frac{x}{2} - 2x\cos\frac{x}{2}\right) + c$

d $-e^{-x}(x + 1) + c$ **e** $-5e^{-4x}\left(\frac{x}{4} + \frac{1}{16}\right) + c$ **f** $x\ln x - x + c$

g $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ **h** $-\frac{1}{25}(\cos 5x + 5x\sin 5x) + c$

i $12\left(x\cos\frac{x}{3} - 3\sin\frac{x}{3}\right) + c$ **j** $\ln\cos x + x\tan x + c$ **k** $\frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + c$

2

a $\frac{2}{15}(3x - 2)(x + 1)^{3/2} + c$ **b** $\frac{2}{15}(3x + 4)(x - 2)^{3/2} + c$ **c** $\frac{2}{15}(3x + 1)(x + 2)^{3/2} + c$

3

a $x\text{Cos}^{-1}x - \sqrt{1 - x^2} + c$ **b** $x\text{Tan}^{-1}x - \frac{1}{2}\ln(x^2 + 1) + c$ **c** $x\text{Sin}^{-1}x + \sqrt{1 - x^2} + c$

4

a $\left(\frac{1}{2}x^2-\frac{1}{4}\right)\text{Cos}^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+c$

b $\frac{1}{2}(x^2+1)\text{Tan}^{-1}x-\frac{x}{2}+c$

c $\frac{1}{4}(2x^2-1)\text{Sin}^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+c$

5

a $\frac{1}{4}$

b $\frac{1}{4}(e^2+1)$

c $\frac{1}{4}(e^2-4)$

d $\frac{1}{4}$

e $\frac{4\pi-\sqrt{2}\pi-4\sqrt{2}}{32}$

f $\frac{1}{2}$

6 $\frac{1}{6}\ln 2+\frac{\pi}{12}-\frac{1}{6}$

7 $\frac{1}{2}[\sqrt{2}+\ln(\sqrt{2}+1)]$

8

a $\frac{x}{2}\cos(\ln x)+\frac{x}{2}\sin(\ln x)+c$

b $-\frac{x}{2}\cos(\ln x)+\frac{x}{2}\sin(\ln x)+c$

c $-\frac{1}{15}(1-x^2)(2+3x^2)\sqrt{1-x^2}+c$

Exercise 6.7.4

1

a $e^x(x^2-2x+2)+c$

b $3\left(\frac{x}{2}\cos 2x+\frac{2x^2-1}{4}\sin 2x\right)+c$

c $\frac{x^4}{4}\log 2x-\frac{x^4}{16}+c$

d $-\frac{e^x}{5}(2\cos 2x-\sin 2x)+c$

e $\frac{2x}{9}\cos 3x+\frac{9x^2-2}{27}\sin 3x+c$

f $-\frac{e^{-2x}}{4}(\cos 2x-\sin 2x)+c$

g $-8\left(x^3\cos\frac{x}{2}-6x^2\sin\frac{x}{2}-24x\cos\frac{x}{2}+48\sin\frac{x}{2}\right)+c$

h $\frac{1}{2}(\ln x)^2+c$

i $2x-2x\ln(3x)+x(\ln(3x))^2+c$

j $-\frac{\cos x}{2}-\frac{\cos 3x}{6}+c$

k $\frac{1}{1+a^4}\left(a^3e^{ax}\cos\left(\frac{x}{a}\right)+ae^{ax}\sin\left(\frac{x}{a}\right)\right)+c$

l $\left(\frac{2x^3}{7}+\frac{4x^2}{35}-\frac{32x}{105}+\frac{128}{105}\right)\sqrt{x+2}+c$

m $\frac{x^4}{4}\ln ax - \frac{x^4}{16} + c$

n $2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4 - x^2} + c$

o $\frac{3}{2}(x\sqrt{x^2 - 9} + 9\ln(x + \sqrt{x^2 - 9})) + c$

p $\frac{1}{2}\ln(x^2 + 4) + c$

q $x - 2\text{Tan}^{-1}\left(\frac{x}{2}\right) + c$

2

a $\frac{\pi^2}{16} - \frac{1}{4}$

b $\frac{\pi}{8}$

c $\frac{1}{2}(e^{2\pi} - e^{\pi/2})$

d $1 - \ln 2 - \frac{1}{2}(\ln 2)^2$

e $\frac{a}{a^2 + b^2}\left(e^{\frac{2a\pi}{b}} + e^{\frac{a\pi}{b}}\right)$

f $e - 2$